Astrophysical Exercises : Formation of Lines in a Stellar Atmosphere

Joachim Köppen, Strasbourg, 2001/2013

Physics Background

Almost all our knowledge about the stars and their chemical composition comes from the interpretation of the absorption lines that are formed in the outer layers of the stars. In this region, called the stellar atmosphere, the density and temperature drop steeply from the high values of the stellar interior to the low ones of interstellar space. From the outside, one is able to look into such a layer down to the depth where the total absorption along the line of sight has become large. One defines an "optical depth" which then becomes unity. The absorption coefficient of the gas varies strongly with wavelength, especially close to ionization thresholds and lines; it depends not only on the chemical composition of the gas, but also on the state of ionization and excitation, which is a function of density and temperature. This means that at different wavelengths we are looking into different layers of the atmosphere. E.g. in the centre of a line the gas absorption is high; so the observable photons come from layers higher up, and hence from cooler layers than the photons far away from the line centre, for which the atmosphere is more transparent. Thus the intensity at the line centre will be lower: an absorption line is formed.

Because the strength and the shape of a line are sensitive to the physical conditions prevailing in the regions where the line is formed, the analysis of the line spectrum of a star yields much important information, but is also quite involved to compute. This involves the radiative transfer, the broadening mechanisms of the line, and the excitation of the ions and the ionization of the elements. In this program, we concentrate on the first two items, but later an extension can be written to deal with the other aspects as well. If one adds more details of line broadening theory, i.e. using a more complicated line profile function, and of atomic physics this program can perform the same tasks as a "professional" one.

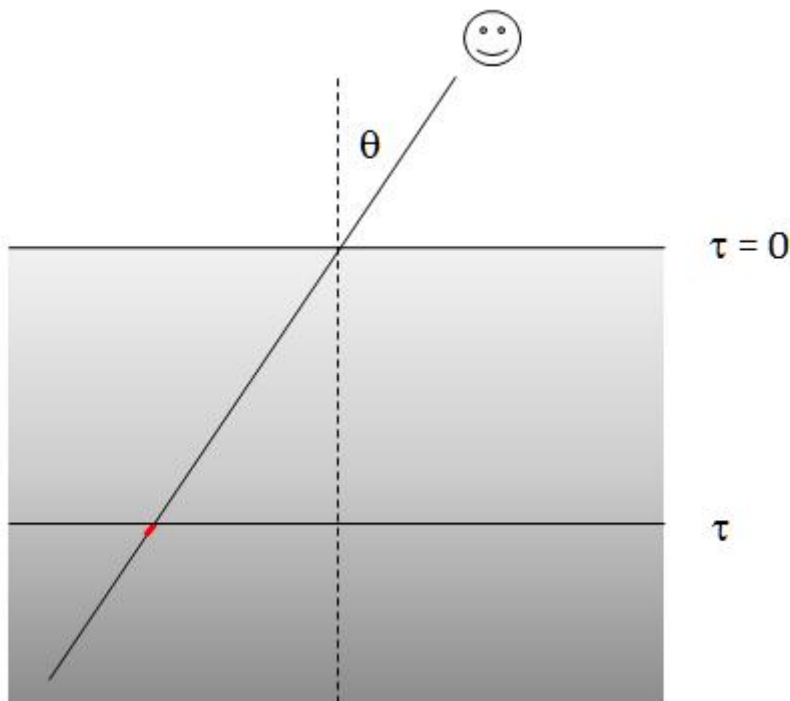## Radiative Transfer in a Plane-parallel Atmosphere

The Equation

In most stars, the atmosphere is but a thin layer; in the Sun it is only a few hundred kilometres thick, negligibly thin compared to the radius of 700000 km. Thus we may neglect the curvature, and represent the atmosphere by a flat (plane-parallel) layer. We shall also assume that the physical conditions do not change in horizontal direction, i.e. there are no star spots, protuberances, granular cells, etc. All these features, which a real star like the Sun has, we shall neglect. Thus we need to know the physical quantities only a a function of the vertical coordinate, which we choose to be the depth t – using the height is equally suitable – starting from the position of the observer. However, rather than using the linear depth, it is more convenient to define the **optical depth** τ, since this is the scale which measures how far the photons travel:

$$\tau(\lambda) = \int_0^t \kappa(\lambda, t')dt'$$

where κ is the local absorption coefficent of the gas at each position. Since it changes with wavelength, so does the optical depth. Thus one selects a standard wavelength λ_0 (for stars like the Sun: 5oo nm) which represents the stratification of the physical parameters in this atmosphere well. Then all quantities (density, temperature, electron pressure, absorption coefficients) are given as functions of this standard optical depth scale τ_0: κ (λ, τ_0) = ...
This can be seen in the **table** for the atmospheric model for the Sun. It is merely a convenience.

The computation of absorption coefficients from atomic and molecular processes is quite complex, so let us assume for the moment that they are given somehow. Then we next compute the **monochromatic** optical depths $\tau(\lambda)$ for all layers at the particular wavelength we are interested in

$$\tau(\lambda, \tau_0) = \int_0^{\tau_0} \frac{\kappa(\lambda, \tau_0')}{\kappa(\lambda_0, \tau_0')} d\tau_0'$$

The geometry of the plane parallel atmosphere

Then the formal solution for the radiative transfer in a planar slab gives the specific intensity $I(\lambda, \theta)$ at the position of the observer (i.e. τ_0 = 0) as

$$I(\lambda, \theta, 0) = I(\lambda, \theta, \tau_0^{max}) \exp\left(-\frac{\tau(\lambda, \tau_0^{max})}{\cos\theta}\right) +$$

$$+ \frac{1}{\cos\theta} \int_0^{\tau_0^{max}} S(\lambda, \tau_0) \exp\left(-\frac{\tau(\lambda, \tau_0)}{\cos\theta}\right) \frac{\kappa(\lambda, \tau_0)}{\kappa(\lambda_0, \tau_0)} d\tau_0$$

The total vertical optical depth of the slab is $\tau_0{}^{max}$, θ is the angle of the line-of-sight with the vertical. The first term describes how any light illuminating the slab from the bottom (intensity $I(\lambda, \theta, \tau_0{}^{max})$) is attenuated by the absorption of the slab. Clearly, if the slab is optically thicker than about 30, one may neglect this contribution altogether. Thus the intensity from a stellar atmosphere is determined by the second term. Each layer contributes to the total radiation by the local {\bf source function}, attenuated by the absorption on the way to the observer, i.e. to the stellar surface. The source function depends on the local temperature, and will be discussed somewhat later.

How to Compute it

The evaluation of the integrals is done numerically. The atmosphere and the physical variables are no longer considered on the continuous optical depth scale τ_0, but at a number of discrete points in the atmosphere. The accuracy and reliability of the results are strongly influenced by a proper choice of this grid. For our problem, the grid points are best placed evenly on a logarithmic scale, such as shown in the model atmosphere results. However, one must always verify that the particular grid chosen is good enough for the purpose.

There are a number of numerical integration prescriptions, with various accuracies, such as Simpson's rule or Gauss' formulae, or more sophisticated ones, usually available from software packages. However, the trapezoidal rule is simple to use, reliable even with strange functions, and accurate, if the function is given at a sufficient large number of grid points, if the grid is fine enough. Then the integral is replaced by a sum

$$\int_a^b f(x)dx = \sum_{i=1}^n f(x_i)w_i$$

over the function given at the n grid points x_i – which need not necessarily be equally spaced – multiplied by the integration weights

$$w_i = \begin{cases} 0.5(x_2 - x_1) & \text{for } i = 1 \\ (x_i - x_{i-1}) & \text{for } 1 < i < n \\ 0.5(x_n - x_{n-1}) & \text{for } i = n \end{cases}$$

If one later wishes to take a different scheme, one changes the prescription for the integration weights. In the depth grid for a stellar atmosphere, it is not necessary to extend the grid to zero and infinity, if the contributions to the intensity by the regions of low optical depth are kept small by choosing a lower boundary $\tau_0(1)$ such that the monochromatic optical depth $\tau(1)$ remains smaller than say 0.01 for all wavelengths. Similarly, the upper boundary $\tau_0(n)$ should be chosen that the atmosphere is sufficiently optically thick (say $\tau > 30$) at all wavelengths, since the contributions of these deeper layers to the emergent intensity go like $\exp(-\tau_0(\lambda,n))$.

Tests

This part of the program described so far can be written first, and should be tested thoroughly, before we take the next steps. Here are some hints how to do it:
- The integration of the monochromatic optical depths should return the standard optical depth scale for all layers, if one computes it at the reference wavelength.

- For a constant $\kappa(\lambda)/\kappa(\lambda_0) = k$ and a constant source function $S \equiv S_0$ one can solve the radiative transfer equation analytically:

$$I = I(0) \exp(-\frac{k\tau_0^{max}}{\cos\theta}) + S_0(1 - \exp(-\frac{k\tau_0^{max}}{\cos\theta}))$$

and the program should reproduce this formula. Eventually the deviations from it give an indication of the accuracy of the program.

- Again with constant absorption coefficient ratio k and with a linear source function $S(\tau_0) = S_0 + S_1 \tau_0$ the radiative transfer has this analytical solution:

$$I(\vartheta) = S_0 + S_1 \cos\theta/k$$

Note that the intensity depends on the angle and opacity ratio k, but not on the opacities.

The Continuum Spectrum

The Source Function

It is a very good approximation in normal stellar atmospheres for most cases to assume **local thermodynamic equilibrium** (LTE), which means that the source function – i.e. the ratio of emission and absorption coefficients – is equal to the Planck function of the local temperature T, given at reference optical depth:

$$S_\lambda(\lambda, \tau_0) = B_\lambda(\lambda, T(\tau_0)) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/(\lambda kT)) - 1}$$

The index λ indicates that the radiative quantities (source function and intensities) are given per unit wavelength interval. If one prefers to measure intensities per unit frequency, one obtains from $B_\nu d\nu = B_\lambda d\lambda$:

$$S_\nu = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/(kT)) - 1}$$

This should be straight forward to program, but if you are not very familiar with Planck's function, you should check the results from the program with what you calculate by 'hand'. Also, it is a good idea just to plot a function to see whether it looks as shown in a textbook. In doing so, you can observe Wien's shift law, which says that the maximum of the curve shifts toward shorter wavelengths when the temperature is increased (λ_{max} T = const.).

If you find it necessary to cut computing time, you may use the various asymptotic forms, e.g. for low frequencies one can expand the exponential function into a Taylor series. The first order approximation yields an expression $B \propto T$ which is commonly used in radio astronomy. Also note that exp(-x) ≈ 0 for x > 50. This saves calling the exponential function unnecessarily, which may take the time of 30 additions or more.

The temperature stratification in the atmosphere could initially be some simple analytical formula $T(\tau_0)$, see below, or eventually taken from a model atmosphere. The data is usually given as a table, so either one searches and interpolates in these tables, or takes the trouble fitting a formula to the data. Do as you like.

Atomic Absorption

Atoms and ions absorb photons over a continuous spectra range, when the energy is high enough to eject an electron in a bound state from the atom, thereby making it a free electron. The absorption coefficient due to this bound-free transition or photoionization can be well described by:

$$\kappa(\lambda) = \kappa_B(\lambda) + \begin{cases} 0 & \text{for } \lambda > \lambda_{ion} \\ \kappa_T(\lambda/\lambda_{ion})^3 & \text{for } \lambda \leq \lambda_{ion} \end{cases}$$

Here κ_B is the background absorption of all the other atoms and processes, κ_T the absorption at the ionization threshold wavelength λ_{ion}. As can be seen, the absorption is largest near the threshold, and decreasing quickly away from it.

Computing the Spectrum

Even if you have carefully tested each part of the program separately, you must not refrain from checking the whole system. Simple models for which strict analytical solutions can be obtained are preferred, such as $S(\tau_0) = S_0 + S_1 \tau_0$\$. Of course this can be done only to a certain extent. For more sophisticated tests, one has to rely on comparison with results obtained with other programs, e.g. from the literature. If the program so far works satisfactorily, you are able to study a few basic things in stellar atmospheres:

For a first inspection of how this absorption feature changes the spectrum of the atmosphere, it is sufficient to assume that the absorption coefficients are independent of height in the atmosphere, set $\kappa_B = 1$ and vary the ratio κ_T/κ_B. If the temperature drops with height, you should see a step at the threshold wavelength, with a lower side at shorter wavelengths. Now turn the temperature upside down, what do you get? If you assume a constant temperature in the whole atmosphere, can you guess what will happen? Why is that so?

The continuum spectrum of stars can be computed roughly, simply by scaling the model temperature structure of the Sun to the particular effective temperature. Since our expression for the continuum opacity is not temperature sensitive, the spectra cannot be compared with real stellar spectra yet. They are essentially the source function of the layer where the radiation comes from. Where is this layer? To study this, plot the expression under the integral in the solution of radiative transfer as a function of τ_0. How does this **contribution function** look like, and what does it tell you? How and why does it differ short- and longward of the ionization threshold?

Now vary the angle θ under which you look into the atmosphere. How does the spectrum change? How do the contribution functions change, and what does it mean? The change of the intensity with angle can be observed in the Sun: Observing at the centre of the Sun's disk ($\theta = 0$) and near the limb ($\theta \approx 90°$) gives information about the temperature stratification in the Sun's atmosphere. The Sun shows a **limb-darkening**. What does this tell you about the temperature gradient?

The Line Spectrum

Line Profile Function and Absorption Coefficient}

To include the effects of atomic or ionic lines on the spectrum, we have to add to the absorption coefficient at each wavelength the contribution by the line:

$$\kappa_{total}(\lambda, \tau_0) = \kappa_{cont}(\lambda, \tau_0) + \kappa_{line}(\lambda, \tau_0)$$

For simplicity, let us deal with a single line only: Then the additional absorption coefficient

$$\kappa_{line}(\lambda, \tau_0) = \frac{k_{line}}{\Delta\lambda_D(\tau_0)} \quad \varphi\left(\frac{|\lambda - \lambda_L|}{\Delta\lambda_D(\tau_0)}\right)$$

depends on the line profile function *phi* of the line centered at wavelength λ_L, and on a factor k_{line} which is proportional to the number of atoms present that are able to absorb line photons. For the moment, let us take this as a given or free variable; later we shall compute it properly.
The line profile function

$$\varphi(x) = \begin{cases} \exp(-x^2) & \text{for } 0 \geq x < 1 \\ \exp(-x^2) + \frac{a}{2\sqrt{\pi}}\frac{1}{x^2} & \text{for } 1 \geq x < 4 \\ \frac{a}{2\sqrt{\pi}}\frac{1}{x^2} & \text{for } 4 \geq x \end{cases}$$

is a function of the wavelength offset from the line centre $x = |\lambda - \lambda_l|/\Delta\lambda_D(\tau_0)$ measured in units of the **Doppler width**

$$\Delta\lambda_D(\tau_0) = \frac{\lambda_L}{c}\sqrt{\frac{2kT(\tau_0)}{m} + \xi_t^2}$$

This width is the wavelength shift due to Doppler effect, caused by the thermal motions of the absorbing atoms in the atmosphere – hence its variation with depth – and by any "microturbulent" motions as well. k is Boltzmann's constant, c the speed of light, and m the mass of the absorbing atom.

The profile function also depends on a parameter $a = \Delta\lambda_{damping}/\Delta\lambda_D$. See below for explanations.

The profile function given above is a simple approximative formula for what is called the Voigt function $H(a, x)$. It is valid for 1E-6 < a < 0.1. Professional programs use better but more complicated and time consuming representations.

Physics Background

Each atom is considered as a driven, harmonic oscillator. Then the absorption coefficient is given by a Lorentz curve that describes the resonance response of the oscillator:

$$\kappa_{line} \propto \frac{1}{(\lambda - \lambda_L)^2 + (\Delta\lambda_{damping})^2}$$

with the radiative damping width $\Delta\lambda_{damping}$ about the centre frequency λ_L. The damping width is determined by how quickly the atom absorbs or emits energy. In the classical picture, it is inversely proportional to the Q-factor, in the quantum mechanical picture it is the inverse lifetime of the atom in the excited state. For common lines – permitted electrical dipole transitions – this lifetime is about 10 ns, so $\Delta\nu_{damping}$ = 1e8 Hz. Thus an optical line at $\lambda_L = 600$~nm has a natural line width of $\Delta\lambda_{damping} = \Delta\nu_{damping} \lambda_L^2/c = 0.0001$ nm.

The thermal motions of the atoms of the gas in the stellar atmosphere cause an additional broadening of the line: If the natural line profile was a delta-function, the Doppler shifts of the

atoms, which pass the line of sight with thermal velocity would make the absorption profile for a volume element of stellar matter to be a Gaussian curve with a width equal to the Doppler width $\Delta\lambda_D$.

The superposition of the two broadening functions results in the Voigt function $H(a,x)$. In our approximative formula one notes that near the line centre Doppler broadening is dominant ("Doppler core") and far away the Lorentz curve takes over ("damping wings"). In the transition part (1<x<4), the formula provides a simple, fairly smooth transition between the two curves.

If you compute the Doppler width and compare it to the natural line width, you find that the latter is much smaller. However, a line can also be broadened by collisions with other atoms, which disturb the atom while it is in its excited state. Thus the life time is effectively shortened, and consequently the line is broadened. For some lines, the parameter *a* is as large as 0.1. Since it is extremely difficult to calculate the broadening coefficient for a line, one often is forced to determine it from a comparison of the observed solar spectrum with a computed one. For our purposes, we may safely assume that *a* is a free parameter. Maybe you can try to adjust it, until you can match the solar spectrum given here.

There are other broadening mechanisms as well: lines of hydrogen are broadened by the Stark effect, i.e. the atoms are disturbed by the change of the electric field, as neighbouring ions pass by.

When people tried to reproduce the lines observed in stars by models, it was noticed that one frequently and systematically falls short to match the widths. Because one can see in the Sun a variety of motions, systematic or turbulent, with various scale lengths, one expects that these motions probably are also present in the atmospheres in other stars, and influence the line spectra. If there are small scale turbulent motions, they would broaden a line just like thermal Doppler broadening. So this can be taken into account simply by adding the microtubulent velocity ξ_t to expression for the Doppler width. Velocities of the order of 1~km/s were found necessary.

At this stage, the program can calculate the profiles of lines from the stellar atmosphere. For the solar spectrum, we may use the model atmosphere of Kurucz to give us the temperature and density stratification, and the absorption coefficient in the continuum. If we allow k_{line}, *a*, and ξ_t to be free parameters, we can try to reproduce some lines observed in the solar spectrum. Please do it. To get experience of how the 3 parameters change the line strength and shape, vary them systematically. When comparing line profiles, you will find it helpful to display and print out the data as a nice plot.

Computing the Curve of Growth

When you investigated the influence of the parameters on the observed line shape and strength, you may have noticed that the main parameters which governs how strong and big the line comes out, is k_{line}. Let us study this more closely.

If one bases a stellar analysis on the interpretation of line profiles, one needs to observe the stars at high spectral resolution, which implies long observing time. However, even if one cannot resolve the line profiles fully, one is able to extract most of the useful information: One considers how much of the continuum light the line is taking away. We define the {\bf equivalent width} by integrating over the wavelength

$$W_\lambda = \int_{-\infty}^{+\infty} \frac{I_{cont} - I(\lambda)}{I_{cont}} d\lambda$$

Then the curve W_l as a function of $k_{line}$$ is the **curve of growth** of the stellar atmosphere. The damping parameter *a* serves as a parameter. Though these curves can be computed for each stellar atmosphere separately, they always have the same character, and actually differ not much, even in details.

The actual curve of growth shown in text books has a slightly different definition: One plots $\log(W_l/\Delta\lambda_D)$ versus $$\log(k_{line}/\Delta\lambda_D)$. This removes the dependence on $\Delta\lambda_D$ and makes the curve more universal.

There are three regimes: for small k_{line} the line is weak, and its equivalent width grows in proportion to an increase of k_{line}. Stronger lines saturate, and W_λ increases only slowly with k_{line}. In this part, the microturbulence ξ_t determines the level of the "plateau". If very strong lines also have strong damping, their equivalent width will increase more strongly, but it depends on *a* as well.
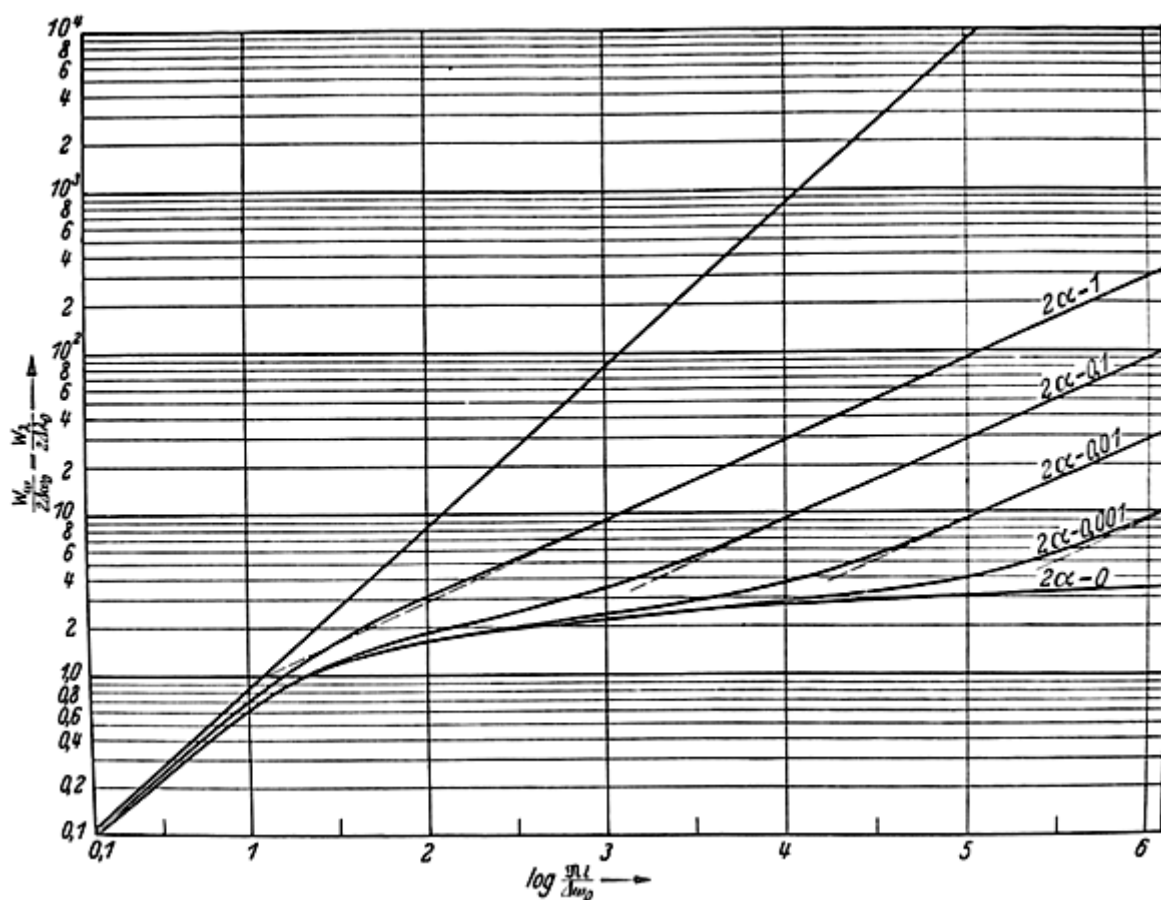
The universal curve of growth (from Unsöld's textbook (1955).

To compute the curve of growth, one has to calculate a large number of line profiles, and integrate them over the wavelength. The continuum intensity I_{cont} is computed in the line centre, and can be taken as constant over the line width. The integration limits should be chosen sensibly. For strong lines they must cover a sufficiently large interval, for weak lines it is sufficient to integrate over a few Doppler widths. Since the integrant may change quite strongly at certain wavelengths, but only slowly at others, you might well try to design an integration routine that automatically adjusts the step width....

Because the curve of growth is a universal one, your results should compare well with those shown in text books. Also, they should not be very sensitive to the temperature or other details of the atmosphere. Is it sensitive to the temperature gradient?

Curve of Growth of Interstellar Absorption Lines

Before one attacks the above problem, it is very useful to do a simpler case: In the spectra of stars and quasars one observes absorption lines of ions and atoms which cannot be present in the stellar atmospheres themselves, and often much narrower than normal stellar lines. They are caused by absorption in the interstellar or intergalactic medium between the star (or quasar) and us. Calculation of their profiles and the curve of growth is much easier than for a stellar atmosphere, as the re-emission (i.e. the source function) can be neglected. Consider the column of gas along the line-of-sight: we view it with $\theta = 0$ and we suppose its total optical depth is τ_λ. The formal solution gives for the observed intensity simply

$$I_{\mathrm{obs}}(\lambda) = I_{\mathrm{star}}(\lambda) \cdot \exp\left(-\tau(\lambda)\right)$$

Note that we do not need to know how κ varies along the line-of-sight. What matters, is the total optical depth. Let us take its value at the line centre τ_0 as a free parameter. If we know or assume the profile function for the atomic absorption (see above) then the above equation permits to compute the observed line profile. How does it change with τ_0?

$$W_\lambda = 2 \int_0^{+\infty} \exp\left(-\tau(\lambda_0 + \Delta\lambda)\right) d\Delta\lambda$$

Show that for a purely Gaussian absorption profile of width *b* one gets:

$$\tau(\Delta\lambda) = \tau_0 \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{(\Delta\lambda)^2}{2b^2}\right)$$

W_λ /b is a unique function of τ_0/b. This permits to compute a universal relation – often called the saturation curve – and use it to determine from the observed equivalent width the optical depth of the intervening medium. When the atomic absorption coefficient for the lines are known from atomic physics, one determines the column densities of the particular atoms, and thus derive the chemical composition of interstellar and intergalactic gas clouds.

Project: Absorption Lines from a Chemically Layered Atmosphere

Here is a small research problem that I'd like you to attack with your program: As you can read in the Rapport du stage by Hélène Perrin (from the class of 2000/01), there are stars whose atmosphere is not chemically homogeneous, and one wishes to know whether some theories for the stratification of the chemistry are correct.

In her stage she investigated how the line profile changes if one assumes that the line opacity k_{line} is not constant (as it approximately is in a normal atmosphere) but that it has a strong stratification. She studied simplified models with

$$k_{line} = \begin{cases} 0 & \text{for } 0 \leq \tau_0 < \tau_a \\ K & \text{for } \tau_a \leq \tau_0 < \tau_b \\ 0 & \text{for } \tau_b \leq \tau_0 < \infty \end{cases}$$

with some value of K. She found that one can easily explain that the line has a certain central depth (her Fig.19) or a certain equivalent width (her Fig.21) etc. for an infinite number of combinations of τ_a and τ_b. So one can explain an observation not only by the line opacity being large only in the upper part of the atmosphere, but equally well by a model where the line opacity is large only in the bottom part, or in a narrow zone in the middle of the atmosphere! What you can do is this
- recalculate her models and check whether you get the same results
- with this simple model one can obtain analytical solutions (see the Rapport). Push this a bit further!
- I have a suspicion that if you plot lines of equal column density (i.e. the integral of k_{line} over all depths, you get very similar contour lines. True? Are the contours the same or are they just similar?
- compute three different stratified models (line opacity at high layers, at low layers, and in the middle) which all should give the same central depression (or equivalent width etc. ...). How do the line profiles differ? What accuracy would be needed to distinguish between these models by observations?

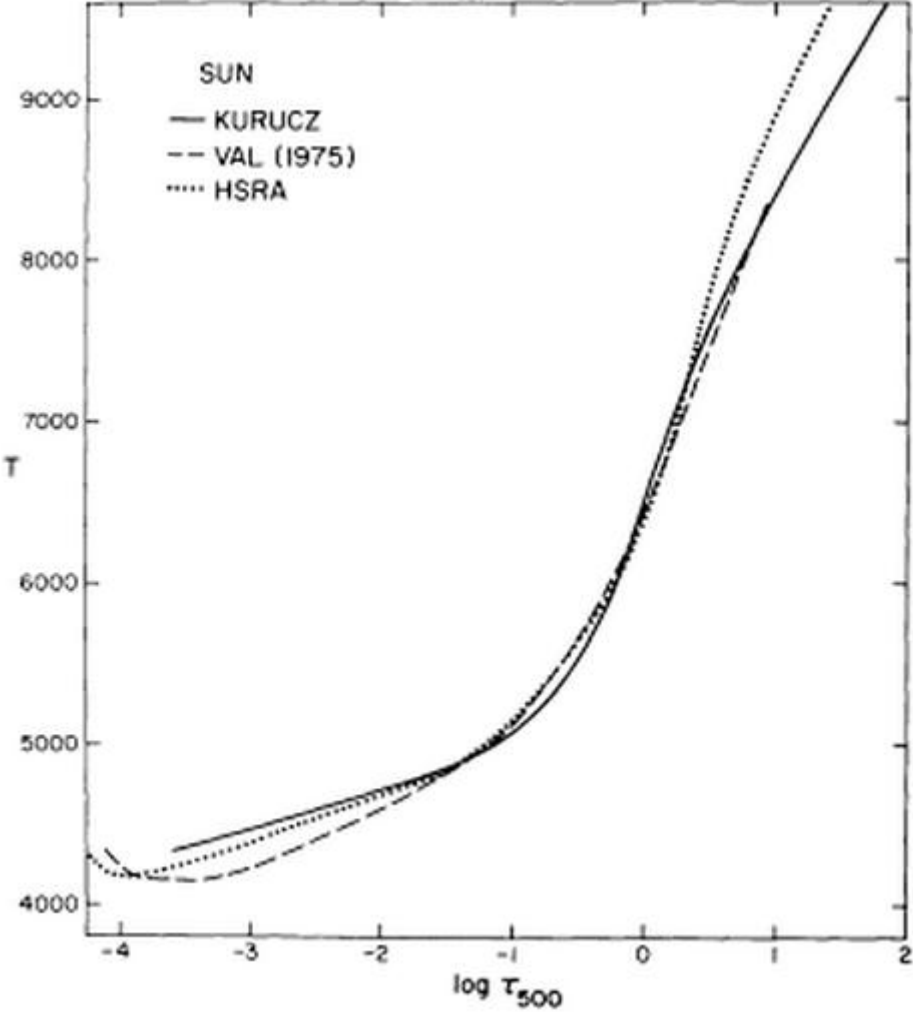Model of the solar atmosphere (R.L.Kurucz, 1979, ApJ.Suppl. **40**, 1)

TABLE 4.—There are 140 solar abundance models, 72 1/10 solar abundance models, and 72 1/100 solar abundance models listed two per page. The heading for each model gives the effective temperature, log surface gravity, log metal abundance relative to solar, and whether the model is convective plus radiative or purely radiative. The 40 depths actually used in the computation are listed, roughly equally spaced in log $\tau_{\rm Ross}$ from -4.5 to 2.0 in steps of $\frac{1}{6}$. Owing to scaling from model to model and to radical temperature corrections, this spacing is not always maintained. To compress the tables, all the variables except temperature are given as logs. The units are cgs for all variables. The columns are mass per unit area, Rosseland optical depth, continuum optical depth at 500 nm, geometric height, temperature, pressure, electron number density, atom number density, mass density, Rosseland mass absorption coefficient, continuum mass absorption coefficient at 500 nm, radiation pressure, radiative acceleration, and the fraction of flux carried by convection. The first values of $\tau_{\rm Ross}$, τ_{500}, and x are defined to be 0. Depths with no convection are listed with 0 convection fraction.

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	MASS	TAU ROSS	TAU 500	X	T	P	NE	NA	RHO	KAPPA ROSS	KAPPA 500	P RAD	ACC RAD	CONV FRAC
1	-1.49997	0.0000	0.0000	0.0000	3054.4	2.9400	10.0976	15.3150	-8.3418	-2.2859	-2.2276	.1716	-.7501	-16.2348
2	-1.36788	-3.6538	-3.5954	6.4024	4329.1	3.0722	11.2206	15.2958	-8.3611	-2.2859	-2.2276	.1719	-1.2426	-22.7006
3	-1.28746	-3.5671	-3.5094	6.6398	4349.8	3.1526	11.2953	15.3740	-8.2828	-2.2167	-2.1650	.1721	-1.2367	0.0000
4	-1.23190	-3.5001	-3.4435	6.7508	4357.6	3.2082	11.3452	15.4288	-8.2280	-2.1698	-2.1210	.1722	-1.2077	0.0000
5	-1.18463	-3.4387	-3.3837	6.8272	4393.3	3.2553	11.3941	15.4724	-8.1845	-2.1268	-2.0864	.1723	-1.2132	0.0000
6	-1.12900	-3.3619	-3.3093	6.9030	4395.3	3.3109	11.4426	15.5278	-8.1290	-2.0802	-2.0413	.1725	-1.1935	0.0000
7	-1.06680	-3.2707	-3.2214	6.9748	4427.8	3.3733	11.5039	15.5870	-8.0699	-2.0252	-1.9934	.1727	-1.1746	0.0000
8	-.99779	-3.1638	-3.1188	7.0430	4447.8	3.4422	11.5680	15.6539	-8.0029	-1.9658	-1.9386	.1730	-1.1572	0.0000
9	-.92358	-3.0435	-3.0035	7.1064	4476.2	3.5164	11.6383	15.7254	-7.9315	-1.9014	-1.8801	.1734	-1.1350	0.0000
10	-.84501	-2.9113	-2.8768	7.1651	4504.3	3.5949	11.7122	15.8012	-7.8556	-1.8336	-1.8176	.1739	-1.1099	0.0000
11	-.76302	-2.7693	-2.7407	7.2192	4535.2	3.6770	11.7896	15.8803	-7.7766	-1.7625	-1.7523	.1746	-1.0816	0.0000
12	-.67862	-2.6199	-2.5974	7.2690	4566.9	3.7614	11.8691	15.9617	-7.6952	-1.6897	-1.6848	.1755	-1.0499	0.0000
13	-.59256	-2.4653	-2.4487	7.3148	4599.9	3.8474	11.9501	16.0446	-7.6123	-1.6158	-1.6157	.1768	-1.0146	0.0000
14	-.50563	-2.3074	-2.2966	7.3569	4634.3	3.9343	12.0322	16.1283	-7.5286	-1.5406	-1.5458	.1784	-.9731	0.0000
15	-.41827	-2.1473	-2.1424	7.3958	4670.7	4.0216	12.1149	16.2121	-7.4447	-1.4642	-1.4755	.1807	-.9245	0.0000
16	-.33063	-1.9855	-1.9866	7.4319	4709.4	4.1092	12.1984	16.2962	-7.3607	-1.3874	-1.4049	.1838	-.8735	0.0000
17	-.24272	-1.8224	-1.8295	7.4656	4751.2	4.1973	12.2826	16.3804	-7.2765	-1.3103	-1.3341	.1880	-.8199	0.0000
18	-.15450	-1.6581	-1.6714	7.4972	4797.4	4.2856	12.3681	16.4645	-7.1924	-1.2326	-1.2630	.1938	-.7637	0.0000
19	-.06584	-1.4924	-1.5122	7.5271	4850.1	4.3742	12.4556	16.5484	-7.1085	-1.1542	-1.1915	.2019	-.7058	0.0000
20	.02281	-1.3262	-1.3527	7.5554	4911.0	4.4628	12.5449	16.6316	-7.0253	-1.0755	-1.1200	.2130	-.6461	0.0000
21	.11131	-1.1598	-1.1931	7.5823	4982.8	4.5513	12.6371	16.7138	-6.9431	-.9961	-1.0482	.2281	-.5847	0.0000
22	.19922	-.9940	-1.0344	7.6078	5069.0	4.6392	12.7325	16.7942	-6.8627	-.9169	-.9765	.2485	-.5217	0.0000
23	.28649	-.8286	-.8763	7.6322	5174.0	4.7265	12.8330	16.8726	-6.7843	-.8362	-.9037	.2757	-.4564	0.0000
24	.37336	-.6626	-.7176	7.6557	5304.1	4.8134	12.9434	16.9487	-6.7082	-.7515	-.8258	.3116	-.3863	0.0000
25	.45895	-.4955	-.5576	7.6782	5467.3	4.8989	13.0715	17.0211	-6.6358	-.6556	-.7359	.3584	-.3029	0.0000
26	.53989	-.3283	-.3973	7.6992	5675.4	4.9799	13.2322	17.0858	-6.5711	-.5321	-.6200	.4177	-.1885	0.0000
27	.61227	-.1594	-.2361	7.7179	5934.0	5.0523	13.4408	17.1388	-6.5181	-.3636	-.4622	.4926	-.0185	0.0000
28	.66993	.0105	-.0749	7.7329	6300.7	5.1099	13.7384	17.1703	-6.4866	-.1224	-.2305	.5842	.2127	-2.2684
29	.71069	.1769	.0840	7.7440	6722.8	5.1508	14.0697	17.1828	-6.4741	.1443	.0328	.6836	.4255	-.9992
30	.74213	.3447	.2442	7.7528	7085.6	5.1821	14.3377	17.1911	-6.4658	.3727	.2496	.7798	.5485	-.5526
31	.76878	.5130	.4026	7.7605	7403.8	5.2087	14.5567	17.1982	-6.4587	.5695	.4297	.8660	.5992	-.3146
32	.79286	.6798	.5580	7.7676	7691.3	5.2327	14.7419	17.2052	-6.4517	.7385	.5853	.9388	.6017	-.1906
33	.81608	.8463	.7130	7.7747	7955.8	5.2560	14.9027	17.2131	-6.4437	.8870	.7235	1.0012	.5885	-.1214
34	.83937	1.0125	.8688	7.7818	8209.4	5.2794	15.0488	17.2222	-6.4347	1.0220	.8523	1.0569	.5708	-.0790
35	.86323	1.1786	1.0254	7.7893	8450.9	5.3032	15.1816	17.2324	-6.4245	1.1511	.9724	1.1078	.5507	-.0598
36	.88811	1.3459	1.1830	7.7972	8686.0	5.3282	15.3051	17.2443	-6.4125	1.2765	1.0873	1.1560	.5354	-.0343
37	.91397	1.5123	1.3398	7.8055	8916.3	5.3539	15.4213	17.2574	-6.3995	1.3957	1.1982	1.2015	.5104	-.0348
38	.94139	1.6788	1.4975	7.8143	9136.9	5.3815	15.5288	17.2729	-6.3840	1.5068	1.3032	1.2451	.4854	-.0118
39	.97057	1.8458	1.6569	7.8239	9369.1	5.4106	15.6371	17.2893	-6.3676	1.6192	1.4118	1.2874	.4573	-.0231
40	1.00167	2.0113	1.8166	7.8340	9578.8	5.4417	15.7333	17.3089	-6.3479	1.7174	1.5097	1.3300	.4612	-.0016