Astrophysics with the Computer: Radio Continuum Emission from the Sun

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1 Astrophysics

When people studied the sky with radio telescopes, they found that the Sun is a very strong emitter of radio waves of various wavelengths. The spectrum is a continnum: for frequencies above about 10 GHz (i.e. wavelengths shorter than 3 cm) it resembles a black body with a temperature of about 10000 K, and the emission is pretty constant. At lower frequencies however, the emission is stronger than that from a black body, and it is also strongly variable: during the solar maximum and especially during periods of solar activity, it can be very strong.

The early studies in the 1950s and 60s established that the radio emission at lower frequencies is due to the hot solar corona. It was found that the radio diameter of the Sun at those frequencies was larger than the optical solar disk, and that the radio Sun was brighter at the rim than in the centre. The origin of the emission is the Bremsstrahlung (or free-free emission) of the electrons in the dilute, hot, and highly ionized coronal gas.

At higher frequencies, the radio image is as large as the optical image, and it resembles more the optical image which shows the photosphere.

Since the absorption coefficient for Bremsstrahlung increases with decreasing frequency, the solar gas is quite opaque at low frequencies and we can see only the outer corona. Going to higher frequencies, the gas becomes more transparent, and the emission comes from deeper layers, until in the infrared and optical we observe essentially the photosphere. Thus, looking at different radio and IR frequencies, one essentially probes the entire structure of the outer atmosphere of the Sun, and the stratification of density and temperature can be obtained.

In this exercise we compute the radio emission of the Sun along a line-of-sight through the corona, transition layer, chromosphere, and the photosphere. Done for different frequencies, we can compute the spectrum. Also, calculation of these intensities for lines of sight at various distances from the optical centre of the solar disk, we can construct radio images of the Sun. We shall use a very simple model for the density and temperature structure of the solar layers. Comparison of the obtained images and the spectrum, we can try to improve the model – or test whether models from the literature fit the data.

2 The Emissivity

The emissivity of a parcel of gas with electron density n_e and temperature T is given by

$$j_\nu = \kappa(\nu, T, n_e) \times B_\nu(\nu, T)$$

the product of the absorption coefficient and the Planck function. It gives the energy radiated per unit volume, per unit time interval, and per unit frequency interval. The latter is indicated by the index ν.

The absorption coefficient depends on frequency, temperature, and electron density:

$$\kappa = n_e^2 \times g_{ff} \times \frac{16\pi^2 e^6}{c\nu^2 (6\pi m_e k_B T)^{3/2}}$$

with a quantum mechanical correction factor (Gaunt factor) for which we can take this approximation formula:

$$g_{ff} = \frac{\sqrt{3}}{\pi} (\ln(\sqrt{\frac{8k_B^2}{\pi^2 m_e}} \frac{t^{3/2}}{e^2\nu}) - 2.5 * 0.577)$$

We note here that the opacity increases like the square of the wavelength!

3 Radiative Transfer along a Line-of-Sight

Consider a line-of-sight on which the opacity and the emissivity are given for every point. If there is no source in the background, the intensity obtained at this line-of-sight and a certain frequency is given by

$$I(p) = \int_0^\infty j(x) \exp(-\tau(x)) dx = \int_0^\infty S(x) \exp(-\tau(x)) d\tau$$

Thus the intensity is the sum of the emissions from each volume element but attenuated by the matter that lies between the element and the observer. The (frequency-dependent) optical depth is given by

$$\tau(x) = \int_0^x \kappa(x) dx$$

The integration coordinate x is taken to be zero at the observer's position, and increases toward the Sun. The second form of the first formula involves the source function $S = j/\kappa$ which in our case is simply the Planck function. Evaluation of the integral can be done in either of the two equivalent forms.

If we want to construct radio images of the Sun, we need to compute the intensity for various offset angles from the centre of the solar disk. It is convenient to consider such a line-of-sight, offset by a distance p from the centre, and to define the coordinate x along the line by in a slightly different way: if we place the zero point at the same distance as the centre of the Sun, there is a simple relation between this coordinate and the true distance r to the centre

$$r = \sqrt{p^2 + x^2}$$

from which we can compute the height h above the photosphere of that volume element. This is useful, because all quantities of our solar model are given as functions of h. The only thing to keep in mind is the sense in which x increases, and hence how to manage the limits of the integration in x.

4 The Solar Model

Below there is a very primitive model for the Solar Chromosphere/Corona. The temperature and the density are given for a number of heights above the photosphere. This gives only a rough representation of what the quiet Sun would look like. The data are taken from various literature models, but values have been added or modified to give a useful starting point.

Please search among the literature, available on the ADS, for better or more complete models! Note that there are models for only certain layers of the Sun, which thus need to be put together, and that there are models for the quiet and active conditions, as well as for the polar and equatorial regions!

height [km]	temperature [K]	density [cm^{-3}]
0	5800	10^{11}
1500	6500	$7\,10^{10}$
2000	10000	$4\,10^{10}$
2010	30000	10^{10}
2030	100000	$4\,10^{9}$
2105	300000	10^{9}
0.005 R_*	700000	$7\,10^{8}$
0.01 R_*	1000000	$5\,10^{8}$
0.1 R_*	1300000	$2\,10^{8}$
0.3 R_*	1700000	$4\,10^{7}$
R_*	2000000	$3\,10^{6}$
2 R_*	2000000	$3\,10^{5}$
3 R_*	2000000	10^{5}
5 R_*	2000000	$3\,10^{4}$

We shall use this kind of model to compute the emissivity at any height. Since the data are given only for some discrete values, we have to interpolate. This can be done with any method you like to use. Of course, the simplest and also safest thing is the linear interpolation: If data y_i and y_{i-1} are given for x_i and x_{i-1}, then the interpolated datum for the value x is

$$y = y_{i-1} + (x - x_{i-1})\frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

If the data change over a several orders of magnitude, it is better to perform the interpolation with $\log(y)$ instead of the linear quantity y. Likewise, if x covers a large range, better interpolate in $\log(x)$.

We shall also make a simplifying assumption: let us neglect the variation of the degree of ionization with height. In principle the corona is more highly ionized than the layers close to the photosphere, but let us simply assume that the electron density is always equal to the density given in the table. In some models you might find the electron density listed separately.

5 Integration along a Line-of-Sight

Computation of the intensity involves the evaluation two integrals along the line-of-sight: first, we have to integrate the absorption coefficient to get the optical depth. Then we need the optical depths to perform the second integration to get the emergent intensity.

We note that zero optical depth in principle is at the observer's position, but we shall neglect any presence of plasma between the Sun and the Earth, mainly because its density can expected to be very small (this might need checking: for frequencies below about 30 MHz the terrestial ionosphere can become opaque! But let us not consider such cases). Therefore we have to choose some distance high above the photosphere to start the integration ... this needs to be tried out!

The next problem is that the density and temperature change dramatically over a small distance close to the photosphere, while in the outer corona, the opacity will vary much less strongly. If we used a constant integration step width, we would require a very small step to cover the deeper atmosphere properly, but that would mean a long computation where most of the time is spent computing the outer corona aith uncessary high accuracy. What is needed, is a variable grid on which the integration is done.

There are two possibilities: We construct by trial and error a grid that is more densely concentrated close to the solar surface. For instance, one could use a logarithmic grid for the height above the photosphere:

$$h_i = h_{\min} \times (h_{\max}/h_{\min})^{(i-1)/(n-1)}$$

One could also take any similar law to make the grid. The other possibility is to construct the grid automatically while the integral is computed. For example, one could demand that the integral should increase by any amount less than a certain fraction ϵ of the integral value so far computed: Then the integration step is determined by a formula like

$$f(x) \times \Delta x < \epsilon \times \int_0^x f(x')dx'$$

If one uses such an approach, we must store the grid points x_i to keep them in order to evaluate on this same grid also the second integral for the intensity. Otherwise, if we use another automatic grid for the second integration, we would need the optical depth values for intermediate points – which could be done by interpolating among the optical depths ...

The best thing is to do the two integrations on the same grid. This means that we have to compute the increments of the optical depth between two grid points. The simplest way is to use the trapezoidal rule:

$$\tau(x_i) - \tau(x_{i-1}) = (\kappa(x_i) + \kappa(x_{i-1})) \times (x_i - x_{i-1})/2$$

You can also choose any other integration method you prefer or like to try out!

Note that the coordinate x is the distance along the line-of-sight, and since the density and temperature (and thus the opacity) are functions of the height above the photosphere, one has to convert the x coordinate into heights h: If the line-of-sight is offset by the distance p from the centre of the solar disk, we have:

$$h = \sqrt{p^2 + x^2} - R_\odot$$

Here we put $x = 0$ at the point on the line-of-sight at the same distance between us and the centre of the Sun. Only for a line-of-sight through the solar disk centre $p = 0$ our coordinate $x = h + R_\odot$ corresponds to the height.

The second integral is computed in a similar way:

$$I = \sum_{i=0}^{n} (S(h_i) \times \exp(-\tau(x_i)) + S(h_{i-1}) \times \exp(-\tau(h_{i-1})) \times (x_i - x_{i-1})/2$$

or, to minimize the time-consuming calls to the exponential function

$$I = \sum_{i=0}^{n} S(h_i) \times \exp(-\tau(h_i)) \times w_i$$

where the integration weights are

$$w_1 = (x_2 - x_1)/2$$

$$w_n = (x_n - x_{n-1})/2$$

and elsewhere

$$w_i = (x_{i+1} - x_{i-1})/2$$

Again, note that the source function depends on the height h.

6 Suggestions for testing

Simple as the trapezoidal rule looks like, you better make sure that your integration routine really implements this rule. Test your routine with simple functions whose integral you can compute analytically. Which class of functions can be integrated exactly by the trapezoidal rule? Check that! In other functions one can check how the error changes as a function of integration step width. Verify that your routine behaves as you should expect for this integration method!

Don't simply crank up the number of points and be content that you get reasonable numbers ... there could be errors lurking beneath that appear only under certain condition and problems ...

Radiative transfer can be checked with the uniform slab: for constant opacity and source function one gets the analytical solution

$$I = S \times (1 - e^{\tau})$$

where τ is the optical thickness of the entire slab. This has two limiting cases: In the optical thin limit, we have

$$I = S \times \tau$$

and in the optical thick case,

$$I = S$$

7 Suggestions of how to proceed

It is a good idea to write the code first with a constant integration step width. One can use a large number of grid points to obtain first results, and only when one has accumulated some experience with the code, one can start optimizing it by developing a variable integration grid.

Evidently, one should start with the line-of-sight through the centre of the solar disk. This permits to compute the radio spectrum of the Sun. Then one deals with offset lines-of-sight, and when this works, one can produce the centre-to-limb variation of the Sun. If one wants, one could use these radial profiles of the radio brightness to construct a (rotationally symmetric) radio imagee of the Sun!

8 Some Questions

- consult the literature for solar observational data ...
- ... and also for more realistic solar models!
- try to match the radio spectrum from about 100 MHz to 30 GHz
- try to match the radio image at some frequency
- the intensity of the radiation which emerges from a slab with temperature and density stratification is approximately equal to the source function at the layer to which the optical depth is unity: $I(\nu) \approx S(\tau(\nu)) = 1)$ ...

- ... if one observes the Sun with a satellite TV receiving system on 11 GHz, one can deduce that at that frequency the Sun appears to have a surface temperature of 11000 K. Does our solar model reproduce this observed value?
- such a receiving system – a simple radio telescope – works in the range 10 to 12 GHz. What variation of the observed surface temperature would be expected over this frequency range?

Note: in radio astronomy, the Planck function can usually be approximated by the Rayleigh-Jeans formula, since $h\nu/k_BT \ll 1$. Therefore, one measures the intensity in terms of temperature:

$$I_\nu = \frac{2\nu^2 k_B T}{c^2} \propto T$$

9 Literature

The book "Numerical Recipes" is available in the MS2 library.

There is a test version of a Java applet - but still without any explanations – available at
`http://astro.u-strasbg.fr/~koppen/radiosun/`