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Gas Flows and the Chemical Evolution of Galaxies: III. Graphical Analysis and Secondary Elements

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ABSTRACT
The general structure of the solutions of the chemical evolution equations of a galaxy undergoing an arbitrary time-dependent accretion of metal-free gas is studied with graphical analysis. The behaviour is determined by the time-evolution of the ratio a of accretion to star formation rate. This gives rise to two distinct regimes: (a) If a decreases monotonically with time, the properties are always between those predicted from the Simple Model and a Linear Accretion Model; the stellar mean metallicity and the abundance ratio of secondary and primary elements ratio are thus restricted to a narrow range of a factor of 2. (b) Lower gas metallicities, larger secondary/primary ratios and stellar metallicities, than expected from the Linear Accretion Models can only be reached if the accretion rate increases rapidly to a value larger than the star formation rate, e.g. by sudden massive accretion of gas into highly evolved gas-poor galaxies.

Key words: galaxies: abundances – galaxies: evolution.

1 INTRODUCTION

The study of the chemical composition of matter in galaxies is one of the most important tools for inferring the physical processes that govern the evolution and structure of galaxies. With the knowledge of these processes and of the production rates of stellar nucleosynthesis, the chemical evolution of a galaxy can be computed. One of the important processes that often needs to be taken into account is the flow of gas into, or out of, the galaxy or some part of it. Gas flows can arise e.g. by the overall collapse of the protogalactic gas cloud, the settling of gas into the disk, or as the consequence of an interaction with another galaxy. Consequently the chemical evolution of the system is altered as compared to a closed-box model, whose analytical solution is useful as a first estimate. But in what way and by how much does its behaviour differ from that of the Simple Model?

To find out what generally valid constraints can be established for the chemical evolution of systems with gas flows has been the aim of this series of studies. Such constraints are useful in understanding both the behaviour of real galaxies, and of numerical models. In Paper I (Edmunds 1990) some simple analytical results on the effects of arbitrary gas flows into, or out of, a galactic system were derived. Here we shall extend the results on the effects of primordial gas flowing into a galaxy from the outside which we shall refer to as *accretion*. This is sometimes called infall or *inflow* in other papers, but we prefer to reserve this term inflow for flows *within* a galaxy, such as radial inflow in a disk, which can have very different chemical effects (Götz & Köppen 1992, Köppen 1994, Edmunds & Greenhow 1995).

In Paper I we found by analytical considerations that the gas metallicity of any system that accretes metal-free gas cannot be larger than expected from the closed-box Simple Model. Similarly, the abundance ratio Z_S/Z of secondary and primary elements cannot be smaller than that of the Simple Model. We shall now take a different approach, and use the fact that *all* the information about all solutions is contained in the differential equations themselves and the initial conditions. A rather convenient way to study the behaviour of the solutions is offered by graphical means: any system of two ordinary differential equations represents a flow (or a field of slopes) in the 2-dimensional space of the independent variables which is uniquely linked to the pattern of streamlines (cf. Hubbard & West 1991). Thus, the possible behaviour of solutions is described as a geometrical pattern. This makes it easy to find steady state and asymptotic solutions as critical points, limit cycles, and funnels, and to identify regions which cannot be reached by certain solutions or can only accessed by others; the local stability of critical points and the global stability of solutions can easily be studied (e.g. Köppen et al. 1996, 1998). It goes without saying that all features found in this way can (and have to) be investigated analytically with the differential equations themselves. This approach can also applied for problems with more than two variables: the equations are solved numerically and their evolutionary tracks are plotted in suitably chosen sub-spaces. In this way, these 'geometrical' or 'graphical' methods – rather than being merely a means to get approximate solutions – offer a powerful approach to study the behaviour of a wide class of problems, without requiring the availability of analytical solutions. On the contrary, they may well provide clues for finding (approximate) analytical solutions.

In the present study, we employ these graphical methods, in combination with analytical considerations, to investigate the evolution of galactic systems with completely arbitrary accretion of metal-free gas. This allows a complete view of the types of possible solutions and their behaviour. We can develop the theorems of Paper I further and in a more general way, and obtain more detailed constraints on the effects on metallicity of gas accretion into a galaxy.

After giving the basic assumptions and equations in Sect. 2, we show the behaviour of the metallicity of the gas (Sect. 3) and the stars (Sect. 4) and of the abundance ratio of secondary and primary elements (Sect. 5).

2 BASIC EQUATIONS

The evolution of the masses of the gas g, long-lived stars s, and (primary) metals $z = Zg$ of a system with instantaneous recycling of the metal rich stellar ejecta is described by

$$\dot{g} = -\alpha\Psi + A \tag{1}$$

$$\dot{s} = \alpha\Psi \tag{2}$$

$$\dot{z} = -Z\alpha\Psi + p\alpha\Psi \tag{3}$$

where α is the fraction of the mass of a stellar generation locked up in long-lived stars and stellar remnants, p the yield, and Z is the fraction by mass of metals in the interstellar medium. **Note** that in the previous papers, s denoted the integrated star formation rate, which differs only by a constant factor ($s_{new} = \alpha s_{old}$); this change simplifies many expressions considerably. In the above equations, we neglect a factor $(1 - Z)$ in the yield, since usually one has $Z \ll 1$. In the numerical solutions this small correction is taken into account, assuming that the yield equals the solar metallicity. The star formation rate Ψ and the accretion rate A of the metal-free gas from outside shall be arbitrary but known functions of time and/or the state of the system. From the last equation one obtains the time change of the gas metallicity Z:

$$g\dot{Z} = p\alpha\Psi - ZA \tag{4}$$

Since the mass of the long-lived stars and stellar remnants will always increase, one may take the star mass s as another measure of time, thereby reducing the equations to:

$$\frac{dg}{ds} = -1 + a \tag{5}$$

$$g\frac{dZ}{ds} = p - Za \tag{6}$$

with the abbreviation $a = A/(\alpha\Psi)$ ($= i/\alpha$ of Paper I). One notices that the constant yield p is not a significant parameter, since the metallicity can always be scaled by dividing it by p. Therefore the chemical evolution is completely described by the behaviour of the ratio a of the accretion rate to the star formation rate. Obviously this could be any non-negative function of s and g; it could be non-monotonic, and even discontinous. This is the most general formulation of an inflow model, with the actual physics in the galaxy only giving tighter restrictions on the form of a.

The current gas mass fraction $f = g/(g+s)$ determines the metallicity of the gas at any time in the Simple Model, by the expression $Z = -p \ln f$. The equation for f is obtained from Eq. 5:

$$g\frac{df}{ds} = -f(1 - a(1-f)) \tag{7}$$

or

$$g\frac{d(-\ln f)}{ds} = 1 - a(1-f) \tag{8}$$

The comparison of any model with the Simple Model leads to the definition of the *effective yield*

$$p_{\rm eff} = -Z/\ln f \tag{9}$$

Obviously, a Simple Model has a constant effective yield, namely the true yield p.

3 METALLICITY OF THE GAS

3.1 Linear Accretion Model: constant a

In the simplest case, the Linear Accretion Model, when a is constant, Eqns. 6 and 8 describe a flow in the $f - Z$ plane, and the evolutionary paths are independent of the gas density:

$$\frac{dZ}{d(-\ln f)} = \frac{dZ/ds}{d(-\ln f)/ds} = \frac{p - Za}{1 - a(1-f)} \tag{10}$$

For a constant value of a, the derivative depends only on the variables f and Z, i.e. the equation describes a pattern of streamlines determined only by the value of the control parameter a. This pattern has the following structural features: The metallicity remains constant whenever

$$Z/p = 1/a \tag{11}$$

and the gas fraction stays constant wherever

$$f = 1 - 1/a \tag{12}$$

The evolution may proceed obeying either of these steady state conditions or even both of them, when possible.

In the following two figures we show in the $\lg(-\ln f) - \lg(Z/p)$ plane how such the system (with a fixed value of a) evolves from different initial conditions. Since the evolution of the gas metallicity is completely determined by the two Eqns. 8 and 6, the evolutionary tracks are proper streamlines, so they do not intersect, and for any value of a the initial values of Z and f completely describe a flow line. Thus the variation of $Z(0)$ and $f(0)$ allows us to probe the general solution, and therefore determine the structure of the general soluion as a function of the control parameter a. The initial values are chosen to cover the $f - Z$ plane in a suitable way, irrespective of how these values could be attained in a galaxy or whether it could actually be done at

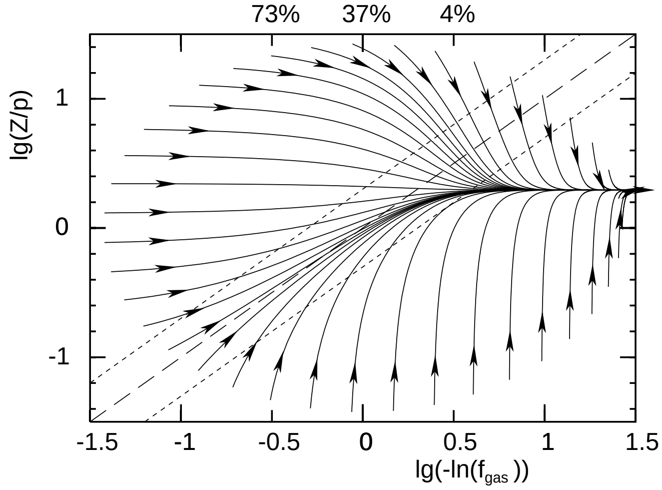

Figure 1. The evolution of chemical models for a constant ratio $a = 0.5$ of accretion and star formation rates. Each model is followed from its initial condition arranged in a circle in the parameter plane. The arrows indicate the direction of time. The dashed lines are the loci of constant effective yield $p_{\mathrm{eff}}/p = 2, 1,$ and 0.5 (from top to bottom). The gas fractions in percent are given on the top.

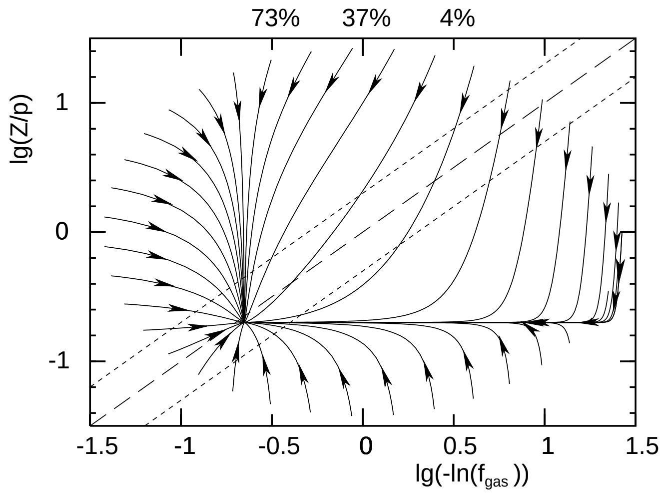

Figure 2. Same as Fig. 2 but for a large value of $a = 5$.

all. From this we obtain complete knowledge of what type of solutions the model equations can produce.

Let us first consider small values of $a < 1$: In Fig. 1 we show how a system with $a = 0.5$ evolves in the $f - Z$ plane from different initial states. Since for small values of a only the first equilibrium condition (Eq. 11) can be met, the streamlines bunch together into a funnel (cf. Hubbard & West 1991) that leads to a constant metallicity $Z = p/a$ while the gas fraction decreases. The centre of the funnel is an asymptotic solution which branches from the evolution of the Simple Model (along the dashed diagonal line) into that for constant metallicity. This solution is in fact the model with 'normal' initial conditions, namely starting with

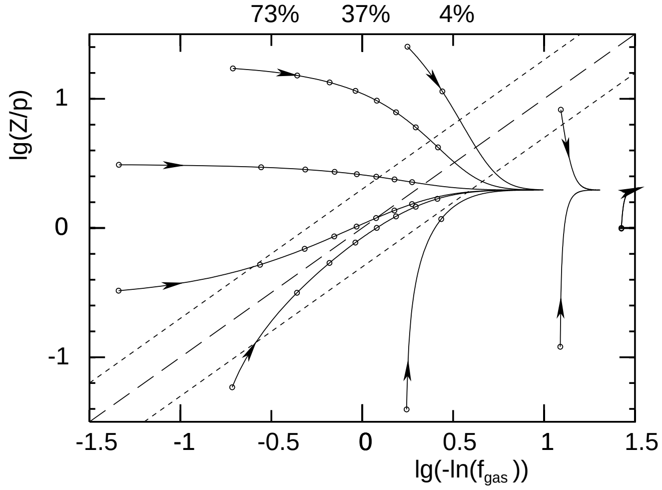

Figure 3. Same as Fig. 1, but showing the total mass as circles (1, 1.1, 1.2, 1.3 ... initial masses).

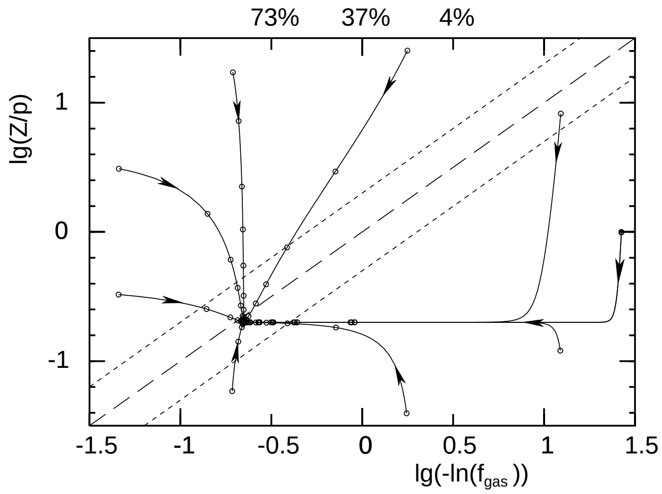

Figure 4. Same as Fig. 2, but showing the total mass as circles (1, 2, 5, 10, 20, ... initial masses).

metal-free gas. The graphical representation gives a good impression that whatever the initial conditions, the model always tends towards the asymptotic solution, i.e. that the equilibrium solution is locally and globally stable. Any small or large perturbation will be followed by an evolution towards the asymptotic solution. In the plot, lines of constant effective yield are parallel to the diagonal. They show that the Linear Accretion Models with small values of $a < 1$ first evolve with the true yield, but for gas fractions smaller than about 37 percent they evolve with effective yields that become smaller with time. Also, one notices that all streamlines that originate in the upper left region (with $p_{\mathrm{eff}} > p$) leave that area, and there is no streamline from wherever that enters that region.

The Simple Model is the limiting case for $a \to 0$: as a decreases, the branch of constant metallicity occurs at higher values, and the funnel is more and more centered about the diagonal.

If the accretion rate exactly equals the gas consumption rate ($a = 1$), the steady state conditions give: $Z = p$ and $f = 0$. The model evolves at constant gas density, turning all accreted gas into stars, the total mass steadily increasing, with the gas fraction thus decreasing as the metallicity tends towards the true yield.

For large values of $a > 1$ both conditions (Eqns. 11 and 12) can be fulfilled, and the steady state is descibed by a single point at high gas fractions and low metallicity, i.e. in less evolved systems. Figure 2 shows the the model with $a = 5$, where all streamlines – and thus all models from any initial conditions – always end up in the equilibrium point $\lg(-\ln f) = \lg(Z/p) = -0.7$. Again, as can be shown analytically, the steady state is evidently stable against perturbations of any magnitude. The effective yield at the steady state point is:

$$p_{\rm eff} = \frac{p/a}{-\ln(1-a^{-1})} = \frac{p}{1+\frac{1}{2a}+\frac{1}{3a^2}+...} < p \qquad (13)$$

For large $a \gg 1$ one gets $p_{\rm eff} \approx p$. So any model with strong accretion ($a > 1$) will end up in a steady state at fixed metallicity, resembling a Simple Model.

The track pattern shows that the equilibrium state is reached in several different ways: if the initial conditions were close to a even less evolved Simple Model i.e. close to the diagonal, the evolution merely follows the diagonal track of the Simple Model further up to higher metallicities. All models that were initially rich both in gas and metals, approach the steady state rather directly, and a weak funnel of decreasing metallicity at the steady-state gas fraction can be identified. The strong bunching of the streamlines in the lower right corner shows another funnel, which we shall term 'late accretion funnel': Whenever strong accretion occurs on to gas-poor models, the metallicity first drops rather quickly as it needs only a small amount of accreted gas to dilute the gas already present in the system. This happens until the steady state metallicity is reached, then the star formation keeps at a level necessary to keep the metallicity constant while the gas accretion continues, and the gas fraction increases. In this manner, streamlines *enter* the region to the lower right of the diagonal, before leaving it on their approach of the steady state.

Figure 2 also shows the only way to enter the region of low metallicity ($Z < p$) and high gas fraction ($-\ln(f) < 1$) by evolution of a system initially consisting of only primordial gas. The area is not accessible by accretion models with small $a < 1$, as they are drawn to their asymptotic solutions with $Z = p/a > 1$. Entry into the region can only be accomplished by *strong* accretion ($a > 1$) onto gas-poor systems, for example highly evolved Simple Models receiving a large late infall of gas.

The two preceding figures show a fundamental feature: the region of high effective yields $p_{\rm eff} > p$ cannot be reached by evolution from anywhere else: no flowline enters it. It can be proven that for all values of a the diagonal – the track of the Simple Model – cannot be crossed except by exiting the high yield region. This is the geometrical equivalent and an extension of Theorem 3 of Paper I which stated that the metallicity in models with metal-free infall (starting with normal initial conditions) cannot exceed the true yield.

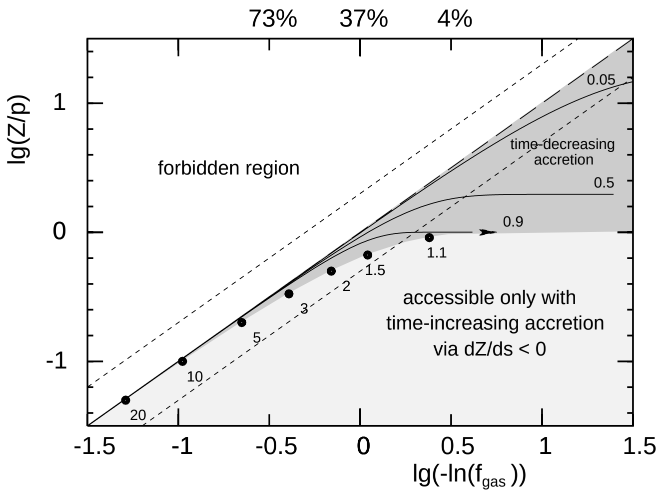

Figure 5. The loci of the steady states in the $\lg(-\ln f) - \lg(Z/p)$ plane. The filled circles show the positions for $a > 1$. The full lines show the evolution of three models with $a < 1$ up to the same time. The dashed lines indicate the relation from the Simple Model with effective yields $p_{\rm eff}/p = 0.5$, 1, and 2.

In Figs. 3 and 4 we show for a small number of streamlines how the total mass of the model increases along the flow line. In all these models, it is assumed that the initial total mass is 1. For the total mass $m = g + s$ one gets from Eq. 5 $dm/ds = a$ which gives with Eq. 7

$$\frac{dm}{df} = -\frac{am}{1-a(1-f)} \qquad (14)$$

which has the solution, provided $f > 1 - a^{-1}$ i.e. for gas fractions larger than the steady state value:

$$m = \frac{1}{1-a(1-f)} \qquad (15)$$

(with $m_0 = 1$ and $f_0 = 1$). For small $a < 1$ the total mass always remains finite: $m < 1/(1-a)$. In Fig. 3 with a rate $a = 0.5$ the total mass never exceeds about 1.8 times the initial mass. Models with high rates $a \geq 1$ will accrete any amount of mass, especially when the steady state is reached (see Fig. 4).

The equation of the streamlines (Eq. 10) is independent of the absolute values of gas or stellar mass fraction, thus independent of the total mass in the system. Therefore, the structures found in the $\lg(-\ln f) - \lg(Z/p)$ plane describe any Linear Accretion Model:

- for $a < 1$ all models tend to the asymptotic solution which goes towards zero gas fraction and a metallicity between that of the Simple Model and the true yield p (Fig. 1).
- for $a > 1$ all models tend to a steady state solution at $f = 1 - 1/a$ and $Z = p/a$ (Fig. 2).
- the steady state loci are on the curve $Z = p(1-f)$ (Fig. 5)
- the region of high effective yield $p_{\rm eff} > p$ cannot be reached by evolution from any initial condition outside that region.

These structures depend only on the value of a, and thus the evolution of more general accretion models can be pieced together from Figs. 1 and 2.

3.1.1 Normal initial conditions

Of particular interest for real galaxies are the models which start with metal-free gas only: $f = 1$ and $Z = 0$. Thus one starts with a metallicity smaller ($Z < p/a$) than any steady state, and the metallicity increases (cf. Eq. 6) towards the steady state value without reaching it. Hence the metallicity increases monotonically, while similarly the gas fraction decreases monotonically. As already mentioned, none of the models can ever enter the region of metallicities larger than that of the Simple Model:

$$Z < -p \ln f$$

On the other hand, no model can cross its steady state locus, so the models are also constrained by

$$Z > p(1 - f)$$

Both conditions are simultaneously fulfilled in the dark shaded region in Fig. 5. The analytical solution for the gas density is found from Eqn. 15 by noting that $g = mf$:

$$g = \frac{f}{1 - a(1 - f)} \tag{16}$$

which is of course only valid for $f < 1 - 1/a$, viz. up to the steady state position. The equation 10 of the streamlines can be integrated directly

$$Z = \frac{p}{a}\left(1 - \left[\frac{f}{1 - a(1 - f)}\right]^{a/(1-a)}\right) \tag{17}$$

also valid only up to the steady state position.

3.2 Models with changing a

The fundamental features of the solutions of the Linear Accretion Model permit us to understand the more general case, i.e. when the ratio a depends on time and the state of the system. We shall consider only changes with s, as dependences on any other variable can be reduced to a suitable dependence on s. Rather than deriving a rigorous mathematical proof (see Paper IV) for each step in what follows, we prefer to describe the typical evolution of models and explain their behaviour on the basis of the (local) properties of the equations as seen in the streamline patterns.

Let us first consider changes of a sufficiently slow for the model to evolve always close to equilibrium. Then it simply follows the movement of the locus of the steady state solution in the $f - Z$ plane:

$$f_e = 1 - 1/a \quad \text{for } a > 1 \tag{18}$$
$$Z_e = p/a \tag{19}$$

which is a point for large $a > 1$ and a horizontal line for small $0 < a \leq 1$. Its position on the curve $Z_e = p(1 - f_e)$ – as a function of a – is depicted in Fig. 5. If a varies very slowly indeed, the system's state would be characterized by a point on this curve.

Suppose now that a decreases with s, starting from a large value (say $a > 5$) close to the corresponding steady

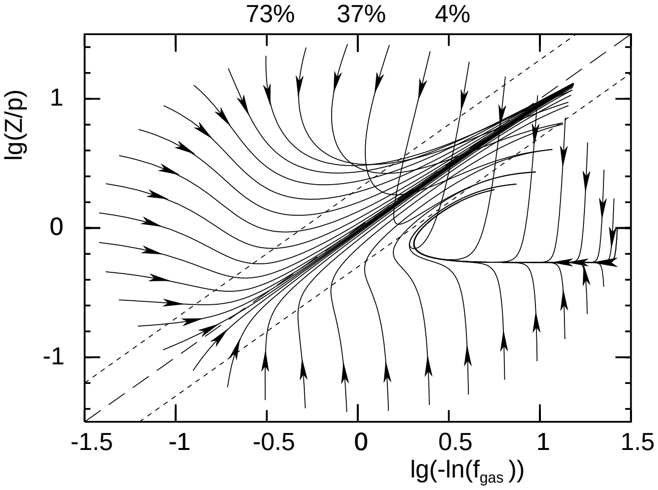

Figure 6. Same as Fig. 2, but with $a(s) = 5e^{-s}$ decreasing with evolution. The total initial mass is $(g + s)_0 = 1$.

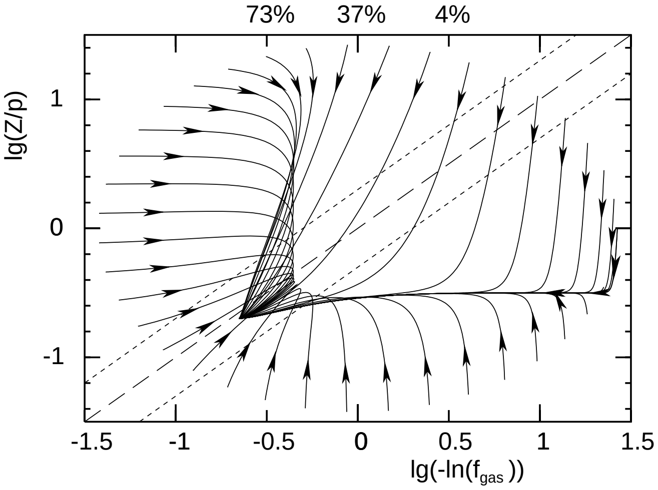

Figure 7. Similar to Fig. 6, but with $a(s) = 5(1 - e^{-s})$ increasing with evolution.

state. As a becomes smaller, the position of the steady state point moves to lower gas fractions and higher metallicities. The system's response is clear from Fig. 2 and Eqns. 8 and 6: f will decrease while Z increases. Note that all streamlines in the quadrant to the lower left of the steady state point will always remain in this quadrant. When a decreases further, the steady state locus moves further up and right, i.e. in the *same sense* as the evolution of the system. So the system's point remains to the lower left of the new steady state, regardless how fast the steady state locus moves about, viz. how quickly a decreases.

Since the model point can never pass through and overtake the steady state, is always lags behind at lower metallicity and higher gas fraction. As long as $a > 3$, the model evolves very close to the diagonal line, that is it behaves like a Simple Model. When a is smaller than about 3, the metallicity increases somewhat less, and the effective yield starts to decrease steadily, to meet $p_{\rm eff} = p/2$ at about $a = 1.2$. When a becomes smaller than 1, the steady state changes from a point to the line of constant metallicity, but as Fig. 1 shows, all models with a lower metallicity will evolve towards the larger equilibrium Z while using up the gas. If a now becomes constant, the models remain at constant $Z = p/a$. But if a continues to decrease further, so does the asymptotic metallicity, and the system's metallicity carries on to catch up with this further increasing limiting value. Depending on how a changes and how rapidly it does so in comparison with the gas consumption, the evolutionary path can be anywhere in the dark shaded region of Fig. 5 bounded by the Simple Model and steady-state loci of the Linear Accretion Models:

$$-p\ln(f) < Z < p(1-f) \tag{20}$$

The upper bound is because the yield of the Simple Model cannot be surpassed by any accreting model (geometrically: that the diagonal line cannot be crossed towards the high-yield region). The lower bound arises because models evolve towards the instantaneous steady state, but cannot overshoot it.

The evolutionary tracks for a model with a monotonically decreasing $a = 5 \to 0$, using some suitable but otherwise arbitrary prescription $a(s)$, are depicted in Fig. 6. Since the behaviour of the system is now described by *three* equations, the tracks are no longer proper streamlines and may cross in any fashion, but nonetheless it shows that the features of the possible solutions are the same as in the Linear Accretion Models: If one starts with a physically 'normal' initial condition ($Z = 0$ and $f = 1$), one follows a track right in the centre of the funnel which is nothing but the Simple Model. Since at late stages, in this particular model $a(s)$ changes more slowly than the gas consumption, the metallicity starts to branch off. Initially metal-poor models evolve also quite close to the Simple Model, branching off at various levels of almost constant metallicity, depending on their initial gas content. In gas-poor models one notices that the gas fraction grows while $\lg(Z/p) = -0.25$ is constant, the value being determined by the value $a(s)$ during this initial phase of evolution. Later, the models seem to branch into steady state solutions for $a < 1$. But one notes that Z keeps on increasing. In this example, we set the initial total mass equal to 1. If one started with a smaller value, or used a higher accretion rate, the funnels around the Simple Model and around the evolution at constant metallicity after accretion onto evolved systems merely become more concentrated, and the latter is shifted to some other Z.

What happens, if the accretion rate increases with time? Suppose that the system is at some point, approaching the current steady state point from the lower left. When a increases, the steady state point will move to higher f and lower Z, i.e. in the *opposite sense* to the evolution of the system; geometrically speaking, it will move towards the point of the current state of the system. If the change of a is sufficiently large and fast, it may move to the lower left of the system point, so that the system is more gas-poor and metal-rich than the current equilibrium. It now must evolve toward the new steady state by increasing the gas fraction and decreasing the metallicity (cf. Eqns. 8 and 6). In this way, the system has effectively overtaken the steady state, and its point has entered the region below the curve $Z = p(1-f)$. In Fig. 7 we show models with $a = 0 \to 5$. Irrespective of the initial conditions, all systems end up in the steady state for $a = 5$. One discerns the funnel at constant metallicity in the initially gas-poor models. The position of the funnel occurs at a higher metallicity than the steady state value, as during this earlier phase a had not yet reached its final value. If one takes a smaller initial total mass or a higher accretion rate, the funnels merely become more pronounced.

The flow pattern allows us to make predictions of the system's response to rapid changes of a: The most extreme case would be a discontinuous jump of a in time. Consider a sudden *decrease* from $a = 5$ to $a = 0.5$ when the system is at equilibrium ($\lg(-\ln f) = \lg(Z/p) = -0.7$): Fig. 1 shows that the evolution simply follows the centre of the funnel for $a = 0.5$. Because on its way towards the new equilibrium the system would have passed anyway through this position, it evolves in the same way – as far as the relation between metallicity and gas fraction is concerned – as a Linear Accretion Model with $a = 0.5$.

On the other hand, consider a sudden *increase* from $a = 0.5$ to $a = 5$ of a quite highly evolved system with a fairly small gas fraction $\lg(-\ln f) = 0.5$ and high $\lg(Z/p) \approx 0.2$ on its way towards steady state. The subsequent evolution (Fig. 2) towards the new steady state takes the model through the region of low effective yield, in fact below the loci of the Linear Accretion Models $Z = p(1-f)$: first the effective yield decreases at roughly constant gas fraction, then $p_{\rm eff} \to p$ increases along the funnel at constant metallicity leading from low to high gas fractions. This shows what happens when a gas-poor model experiences a sudden gas accretion: first, the metallicity decreases until it reaches the value corresponding to the new steady state, then the gas fraction is increased, and the new equilibrium is attained.

Whether the change in a, be it by its dependence on time, gas mass, or stellar mass, is slow or rapid, can be assessed by inspecting how much of the changes of the derivatives of f and Z are due to a change in a:

$$\begin{aligned} \frac{d^2 f}{ds^2} \quad = \quad & \frac{da}{ds}\frac{f(1-f)}{g} \\ & -\frac{df}{ds}\frac{1-a(1-f)+af}{g} \\ & -\frac{f(1-a)(1-a(1-f))}{g^2} \end{aligned} \tag{21}$$

giving a condition for the smallness of the change in a:

$$\left|\frac{da}{ds}\right| \quad < \quad \frac{2af^2}{g^2}\cdot|1-a(1-f)| \tag{22}$$

which can be met most easily if the R.H.S. is large: This occurs for high gas mass g and if the system is far away from the steady state, where the second term would vanish.

Likewise, the change of the metallicity due to changes in a gives:

$$\frac{d^2 Z}{ds^2} = -\frac{da}{ds}\frac{Z}{g} - \frac{p-aZ}{g^2} + \frac{p-aZ}{g^2}(1-a) \tag{23}$$

which leads to the condition for the smallness of the change in a as

$$\left|\frac{da}{ds}\right| \quad < \quad \frac{a}{g^2}|p-aZ| \tag{24}$$

Again, this condition is met for high gas mass g or far away from steady state $Z = p/a$. Under these circumstances, one can expect that the system evolves faster than the changes in a.

What would be a realistic model? At the beginning of the evolution of a volume in a galaxy, this volume is filled with some gas. Due to the collapse of the protogalactic cloud other gas may flow into the volume, and if there is little or no star formation, we may expect the ratio a to be rather large. Thus the system would evolve like a Simple Model. With the rising gas in the volume the star formation increases, and a decreases with time. The effective yield would still be close to the true yield. This will continue to be so, if the gas inflow fizzles out to a level lower than the star formation rate. But if the accretion remains at some fraction a of the star formation rate, the gas is consumed at constant metallicity $Z = p/a$. Depending on the behaviour of a, Z may further increase, stay constant, or even decrease. During this phase, the model may no longer evolve so quickly as to follow the changing steady state, but it can only remain in the region accessible for models with decreasing accretion rate. If a and thus Z remains constant, the effective yield obviously decreases with time, tending towards zero.

The metallicity cannot increase indefinitely, because eventually the gas return from long-lived stars – which is neglected here – becomes important.

3.3 Properties of Accreting Models

The evolution of gas fraction and gas metallicity in models with arbitrary prescriptions of accretion of metal-poor gas is described by the pattern of the streamlines (Eq. 10) which depends only on the parameter a. The structures found in models with constant values of a determine the basic behaviour of models in which a is an arbitrary function of time. There are two important borders:

- the Simple Model $Z = -p\ln(f)$ can be crossed in only one sense: when exiting the region of high effective yield
- the locus of Linear Accretion Models $Z = p(1 - f)$ can be crossed by any model in the sense of increasing the effective yield, but in the opposite sense this is only possible by strong accretion $a > 1$ onto gas-poor systems.

Thus, in the $f - Z$ plane one may distinguish three regions (Fig. 5):

- $Z > -p\ln(f)$: metallicities larger than those expected from the Simple Model cannot be reached by evolution from any initial condition outside that region and for any type of evolution of a
- $-p\ln(f) > Z > p(1 - f)$: in the region of the Linear Accretion Models, the models with monotonically decreasing a can be found, but also those whose increase in a is not large or rapid enough to let it enter the region of low effective yields
- $p(1-f) > Z$: the area of low effective yields can only be reached by strong $a > 1$ accretion increasing rather quickly in gas-poor and metal-rich models

Thus, the region designated in Fig. 1 of Paper I as accessible by infall is actually composed of the two latter regions.

4 STELLAR METALLICITY

The mass weighted stellar metallicity defined as

$$\langle Z \rangle := \frac{1}{s}\int_0^s Z(s')ds' \tag{25}$$

is a function of s and obeys the differential equation

$$s\frac{d\langle Z \rangle}{ds} = Z - \langle Z \rangle \tag{26}$$

which has a steady state solution $Z = \langle Z \rangle$. Note that this condition does not explicitely depend on the accretion a. The evolutionary tracks in the $\langle Z \rangle - f$-plane are proper streamlines

$$\frac{d\langle Z \rangle}{d(-\ln f)} = \frac{g(Z - \langle Z \rangle)}{s(1 - a(1 - f))} = \frac{f(Z - \langle Z \rangle)}{(1 - f)(1 - a(1 - f))} \tag{27}$$

which depend only on the accretion ratio a, because the gas metallicity is a function of only a and f (cf. Eq. 10). The streamline pattern is independent of s and g, thus of the total mass of the system.

Figure 8 shows that in Linear Accretion Models with a small value $a < 1$ the stellar metallicity follows that of the Simple Model very closely:

$$\langle Z \rangle = \langle Z \rangle_{SM} = p(1 - f\frac{(-\ln f)}{1 - f}) < p$$

(from Eq. 35 of Paper I). The steady state is approached in the limit $f \to 0$, with $\langle Z \rangle \to p$. Models with large values $a > 1$ (Fig. 9) have a steady state solution characterized by the three conditions

$$f_e = 1 - 1/a \text{ and } \langle Z \rangle_e = Z_e = p/a \tag{28}$$

Thus, the locus where the steady states of all Linear Accretion Models lie, is given by:

$$\langle Z \rangle_e = Z_e = p(1 - f_e) \tag{29}$$

As can be seen in the figures, at low gas fractions, this locus merges with the Simple Model, while for large values of $f \to 1$ one finds

$$\langle Z \rangle_{SM} \to 2p(1 - f) = 2\langle Z \rangle_e$$

using $-\ln(f) \approx (1 - f)(1 + (1 - f)/2 + (1 - f)^2/3 + ...)$. Thus, the region between the steady states and the Simple Model is quite limited.

In the figures one can distinguish structures and regions in the $f - \langle Z \rangle$ plane corresponding to those found earlier in the $f - Z$ plane for the gas metallicity: the region of $\langle Z \rangle$ smaller than expected from the Simple Model is 'forbidden' as all streamlines exit it. The region of stellar metallicities larger than those of the steady states cannot be entered by many model with small $a < 1$. It is only the strong accretion $a > 1$ onto systems with small gas fractions that access this region. As shown in Fig. 10, models with accretion decreasing monotonically in time evolve in the region between the Simple Model and the steady states. Both curves converge at low gas fractions, the stellar metallicity of all these models eventually evolves like a Simple Model. Finally, a rapid increase of a to values larger than 1 (Fig. 11) causes the models to approach the final steady state, via the region where the stellar metallicity is larger than expected from Linear Accretion Model.

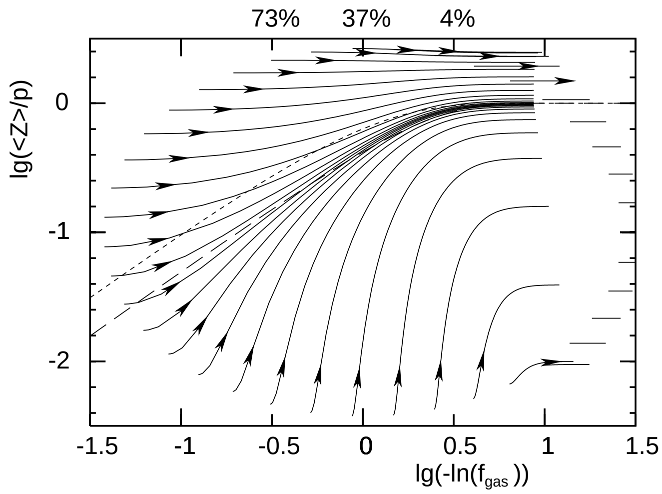

Figure 8. The evolution of the stellar metallicity $\langle Z\rangle/p$ gas metallicities with the gas fraction f in a model with $a = 0.5$. The initial gas metallicity is always set to $Z = \langle Z\rangle$. The dashed curve is the solution of the Simple Model, the short dashed curve is the locus of the Linear Accretion Models.

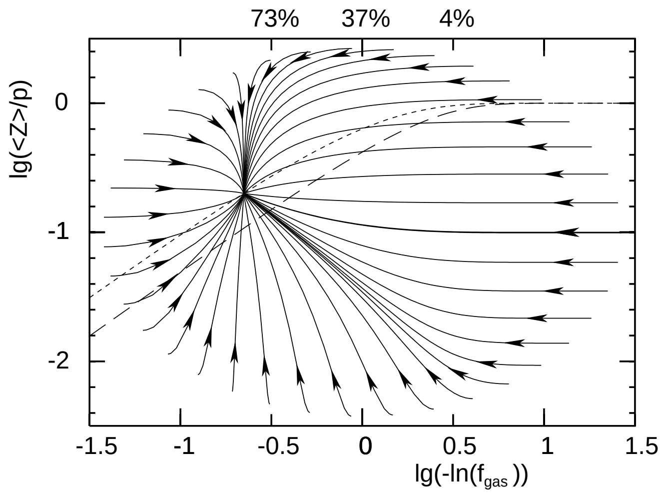

Figure 9. Same as Fig. 8, but for a model with $a = 5$.

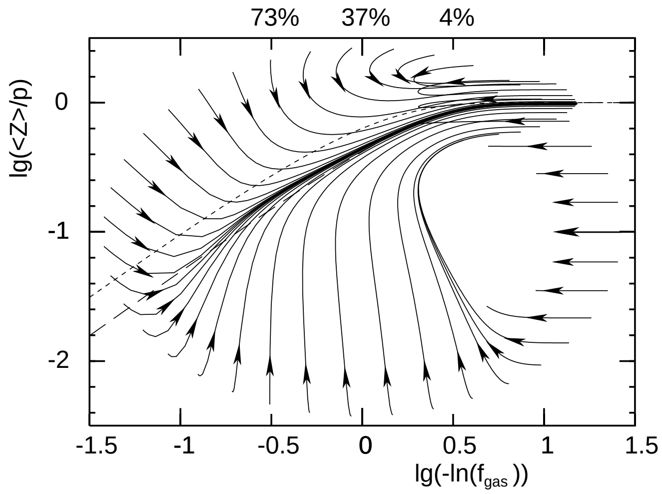

Figure 10. Same as Fig. 8, but for a model with $a(s) = 5e^{-s}$ decreasing with time. The initial gas metallicity is always taken to be $Z = 0$.

One notices that the condition for the 'forbidden' region seems to be somewhat violated in the models shown here, as streamlines enter this region at large gas fractions. The reason this happens is that the initial state of the model is not uniquely defined by the initial values of f and $\langle Z\rangle$ only, but one must also specify the initial gas metallicity Z. As this can be arbitrarily chosen, e.g. it is set equal to the initial $\langle Z\rangle$ as in Figs. 8 and 9 or set equal to zero (in Figs. 10 and 11), the orientation of the streamlines may vary in the upper left-hand portion of the plots. In physically realistic models, starting with only metal-free gas and no stars, this inconsistency does not arise; this illustrates the effects of (physically) inconsistent initial conditions.

Because the mean stellar metallicity in Linear Accretion Models converges for small gas fractions to the Simple Model, the stellar metallicity are less sensitive to effects of accretion than the gas metallicity, which does not have such a limiting behaviour.

5 SECONDARY ELEMENTS

The equations for the gas metallicities of a purely primary (Z) and a purely secondary (Z_S) element and their ratio $R := Z_\mathrm{S}/Z$ are

$$g\frac{dZ}{ds} = p - aZ \tag{30}$$

$$g\frac{dZ_\mathrm{S}}{ds} = p_s Z - aZ_\mathrm{S} \tag{31}$$

$$gZ\frac{dR}{ds} = p_s Z - pR \tag{32}$$

They show that there is a steady state solution obeying:

$$Z = p/a \text{ and } Z_\mathrm{S} = p_s Z/a \tag{33}$$

Then the ratio is $R = p_s Z/p$, which means that in steady state $R/Z = p_s/p$ is independent of the accretion parameter a. Thus for all values of a, the equilibrium is always a point in the $R-Z$-plane. In the usual plot of $\lg(Z_\mathrm{S}/Z)$ vs. $\lg(Z)$ one finds proper streamlines

$$\frac{d\lg R}{d\lg Z} = \frac{1}{R}\frac{p_s Z - pR}{p - aZ} \tag{34}$$

whose pattern does *not* depend on s or g, and therefore not on the absolute mass of the system under consideration. It seems to depend on a and on the yield ratio p_s/p, which we had set equal to unity, but they are completely scaled out by relating the tracks to the steady state point (i.e. shifted in the log–log plot):

$$\frac{d\lg(Ra/p_s)}{d\lg(Za/p)} = \frac{1}{(Ra/p_s)}\frac{(Za/p) - (Ra/p_s)}{1 - (Za/p)} \tag{35}$$

Thus the two Figs. 12 and 13 show the same structure, merely seen in two views shifted against each other.

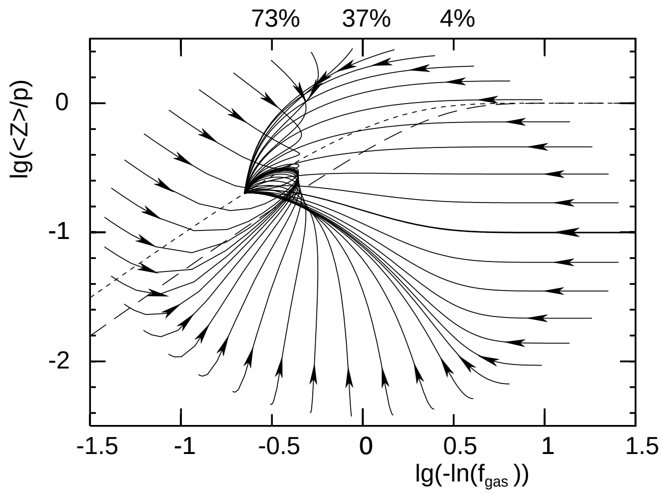

Figure 11. Same as Fig. 10, but for a model with with $a(s) = 5(1 - e^{-s})$ increasing with time.

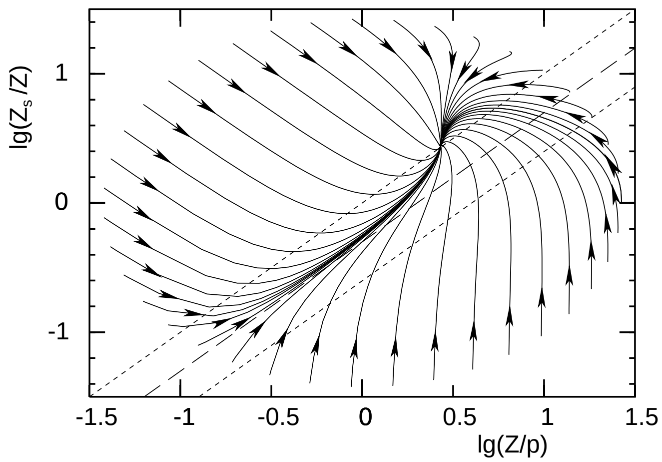

Figure 12. The evolution of the ratio of secondary to primary gas metallicity as a function of the primary metallicity, for a model with constant ratio $a = 0.5$ of accretion and star formation rates. Each model is followed from its initial condition arranged in a circle in the parameter plane, always taking $f = 1$ for the initial value of the gas fraction. The long dashed lines are the expectations from the Simple Model ($p_s/2p = 1$), the short dashes indicate multiplication by a factor 2 and 0.5.

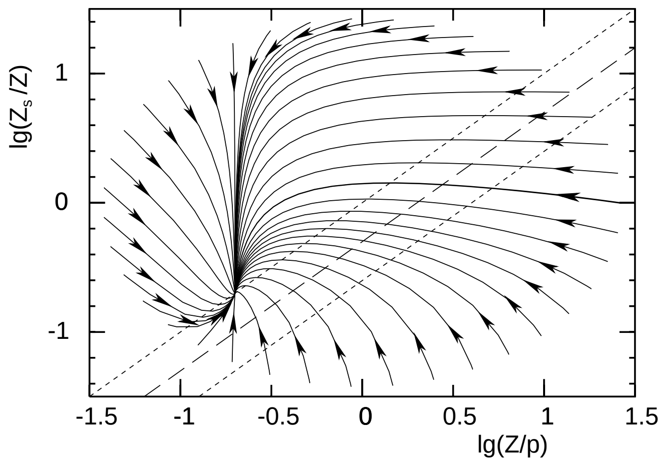

Figure 13. Same as Fig. 12, but for a value $a = 5$.

Linear Accretion Models (i.e. with constant values of a) just evolve along streamlines. In the patterns shown in Figs. 12 and 13 the flowlines have horizontal tangents where $R = Zp_s/p$ (on the upper short dashed line), and vertical tangents for $Z = p/a$. These two loci intersect for all values of a; the pattern illustrates that this point of intersection is an attracting node, which is confirmed by linear stability analysis.

Models with normal initial conditions – $Z = Z_S = 0$ and $f = 1$ – first evolve like the Simple Model, and then approach the steady state in the centre of a funnel. For this asymptotic curve we derive the analytical solution by integrating the equation $dZ_S/dg = (p_s Z - Z_S a)/(g(a-1))$ by using the analytical solution for Z (Eqn. 17) and the integrating factor $g^{a/(a-1)}$ to give

$$Z_S = \frac{p_s}{a}\left(Z + (\frac{p}{a} - Z)\ln(1 - \frac{Za}{p})\right) \tag{36}$$

which shows that for a given values of a the secondary metallicity depends explicitly only on the primary metallicity.

When the accretion changes with time, models behave in a manner analogous to what is found for the gas metallicity. If the changes occur very slowly, the models try to follow the changing position of the relevant steady state which is always a single point: $R = Zp_s/p$. Models where a decreases in time evolve towards the upper right, while the steady state point moves in the same direction, of increasing Z. If a stops at some value, the model will end up in the corresponding steady state, if a continues to decrease towards 0, the Simple Model solution is approached (Fig. 14). In either way, ratio R stays within the area bounded by the Simple Model and the steady states of the Linear Accretion Models, as the latter cannot be overtaken.

But if the accretion increases so rapidly that the new steady state is at a lower Z than the current metallicity, the models evolve by entering the region of Z_S/Z ratios larger than expected from the Linear Accretion Models, as can be seen from the streamlines in Fig. 13. Models with time-increasing a (Fig. 15) approach the final steady state by a loop of this type, especially models which started at a highly evolved state.

In analogy to the gas and stellar metallicities, one discerns the corresponding regions in the $Z - R$ plane (Fig. 16): the line of the Simple Model is only crossed towards the upper-left, thus the region with Z_S/Z ratios smaller than the Simple Model is forbidden. The line of the steady states can only be crossed towards the upper-left, if the accretion increases sufficiently fast and strongly to have $dZ/ds < 0$ in Eqn. 6. Finally, models with accretion decreasing monotonically in time, as well as those with a increasing slightly

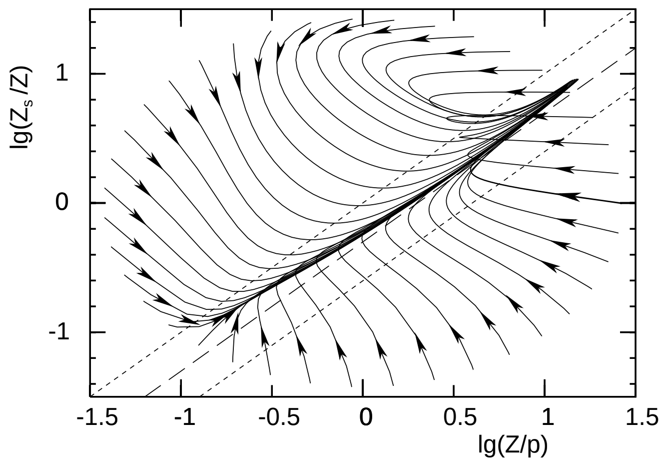

Figure 14. Same as Fig. 13, but for a model with a decreasing from 5 to 0.

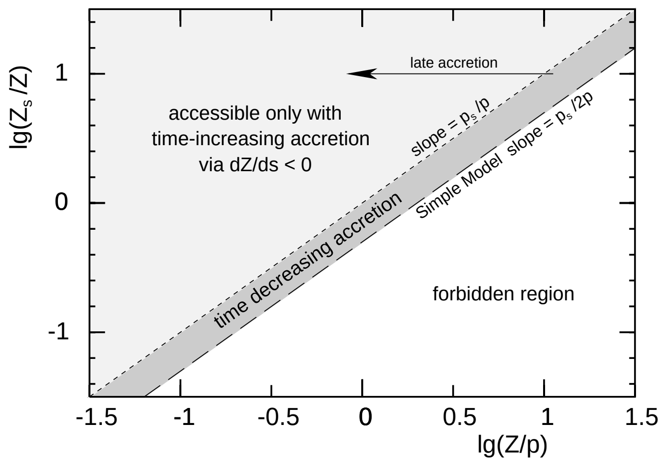

Figure 16. The types of behaviour of accretion models in the secondary/primary vs. primary metallicity diagram.

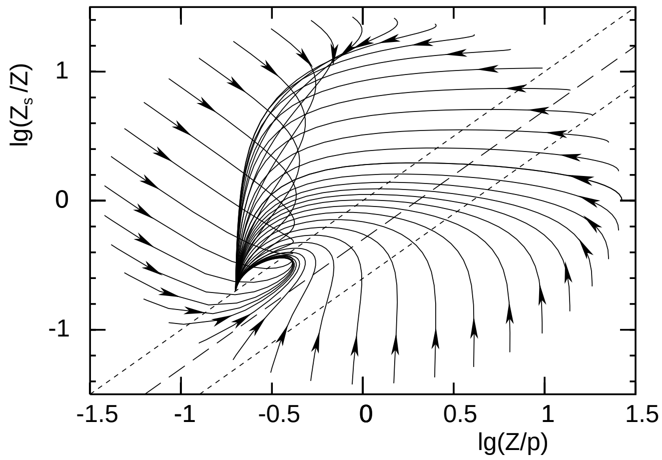

Figure 15. Same as Fig. 13, but for a model with a increasing from 0 to 5.

or slowly, are restricted to the region between the Simple Model and the steady states:

$$Zp_s/p > Z_S/Z > Zp_s/2p \tag{37}$$

which is a rather narrow strip only 0.3 dex wide in the $Z_S/Z - Z$ diagram.

6 ACCRETION OF GAS CONTAINING METALS

If the accreted gas contains metals at some constant abundance value Z_f, one can quite easily discuss the properties of the metallicity evolution in terms of the solutions for metal-free accretion. As shown and discussed in Paper I (Theorem 10), the transformation $Z \to Z - Z_f$ leaves the form of the equation for the gas mass and thus the solutions unchanged. Thus the steady states of the Linear Accretion Models are simply shifted to $Z = Z_f + p/a$ and the curve of the steady state loci in Fig. 5

$$Z = Z_f + p(1 - f) \tag{38}$$

now turns into a horizontal line ($Z = Z_f$) for large gas fractions.

In the limit $a \to 0$ one obtains the Simple Model with a non-zero initial metallicity with its analytical solution

$$Z = Z_f - p\ln(f) \tag{39}$$

i.e. the curve of the Simple Model in Fig. 5 is also bent into the horizontal line $Z = Z_f$ for large gas fractions. Note that this metallicity remains always larger than that of the steady state (Eqn. 38) for a Linear Accretion Model having the same gas fraction. While the dark shaded region in Fig. 5 is deformed to curve into the $Z = Z_f$ horizontal line, the sequence of the three regions remains the same. The same holds true for the pattern of the streamlines. This means that in the early phase one has $Z \approx Z_f$, while for metallicities well above Z_f the results of this paper can be applied directly.

The equation for the mean stellar metallicity is not changed, and so is the steady state condition

$$\langle Z \rangle = Z = Z_f + p(1 - f) \tag{40}$$

which now refers to a shifted metallicity.

In analogy to a primary element, the abundance enhancement of a secondary element over its abundance Z_{Sf} in the accreted gas is given by

$$g\frac{d(Z_S - Z_{Sf})}{ds} = p_s Z - (Z_S - Z_{Sf})a \tag{41}$$

The steady state condition now involves the ratio of the abundance differences

$$\frac{Z_S - Z_{Sf}}{Z - Z_f} = \frac{p_s Z}{p} \tag{42}$$

which gives

$$R_{\rm LAM} = \frac{Z_{\rm S}}{Z} = \frac{Z_{{\rm S}f}}{Z} + \frac{p_s}{p}(Z - Z_f) \tag{43}$$

In the beginning, when $Z = Z_f$, the ratio is equal to that of the accreted gas $R = Z_{{\rm S}f}/Z_f = R_f$. Depending on the yields, R may go through a minimum as long as the first term decreases faster than the second increases, but eventually the ratio will increase linearly with Z.

The solution of the Simple Model with non-zero initial metallicities is

$$Z_{\rm S} - Z_{{\rm S}f} = \frac{p_s}{p}\left(Z_f(Z - Z_f) + \frac{1}{2}(Z - Z_f)^2\right) \tag{44}$$

and so

$$R_{\rm SM} = \frac{Z_{{\rm S}f}}{Z} + \frac{p_s}{2p}(Z - Z_f)(1 + \frac{Z_f}{Z}) \tag{45}$$

Comparison with the Linear Accretion Models at the same metallicity Z gives

$$R_{\rm SM} - \frac{Z_{{\rm S}f}}{Z} = \frac{1}{2}(1 + \frac{Z_f}{Z}) \cdot (R_{\rm LAM} - \frac{Z_{{\rm S}f}}{Z}) \tag{46}$$

which shows that as $Z \geq Z_f$ is always true, one always has $R_{\rm SM} \leq R_{\rm LAM}$. For large metallicities $Z > Z_f$ it approaches $R_{\rm SM} = R_{\rm LAM}/2$, i.e. the relation for metal-free accretion. Thus, in Fig. 16 the lines denoting the Simple Model and the Linear Accretion Models will merge towards low metallicities into a single point at $(Z_f, Z_{{\rm S}f}/Z_f)$ – the initial situation. Note that the secondary/primary ratio R may well decrease during the early phases of evolution, depending on the yields and the initial abundance ratio. Therefore the curves may drop in R before joining the metal-free solutions. But the sequence of the three regions in Fig. 16 remains the same, and this holds also for the pattern of the streamlines.

If the accreted gas is not metal-free, changes to the steady state solutions, the streamline pattern, and the three regions of characteristic behaviour occur only at (the low) metallicities close to that of the accreted gas, and can be easily described. They do not change the basic features of the solutions found for metal-free accretion.

7 CONCLUSIONS

The chemical evolution of galaxtic systems with an arbitrary time-dependent accretion of zero-metallicity gas is investigated with graphical techniques to show the possible types of solutions for the metallicity of gas and stars, and the abundance ratio of secondary and primary elements. Assumption of the instantaneous recycling approximation permitted the elimination of explicit time depence as well as of the dependence on a particular prescription for the star formation rate. Under these circumstances, the type of the solutions is controlled by a single parameter: the ratio a of the instantaneous rates of accretion and star formation. It is this ratio and how it changes with time that determines the evolution of a model.

In the relations between gas fraction f and gas metallicity Z, f and stellar mean metallicity $\langle Z\rangle$, and primary and secondary element abundances in the gas (Z, $Z_{\rm S}$) three regions of characteristic behaviour are distinguished:

- gas metallicities *higher* than that of the Simple Model

$$Z > -p\ln(f),$$

smaller stellar metallicities

$$\langle Z\rangle < p(1 - f(-\ln(f))/(1 - f)),$$

or secondary-primary ratios *smaller* than in the Simple Model

$$Z_{\rm S}/Z < Zp_s/2p$$

cannot be achieved by evolution from normal initial conditions by metal-poor accretion, whatever its time-dependence. Galaxies that exhibit at least one of these properties, cannot be explained by this type of model, and other physical scenarios must be considered.

- properties between the Simple Model and the steady states of the Linear Accretion Models:

$$-p\ln(f) > Z > p(1 - f)$$

$$p(1 - f(-\ln(f))/(1 - f)) < \langle Z\rangle < p(1 - f)$$

$$Zp_s/2p < Z_{\rm S}/Z < Zp_s/p$$

are obtained with models having an accretion decreasing monotonically with time, as well as accretion increasing slowly in time and/or only slightly, so that $Z(t) < p/a(t)$ is always obeyed. Since in a physically realistic model of a galaxy forming out of a collapsing protogalactic gas cloud one can expect the accretion into the disk to decrease with time, the chemical properties of such models are quite strongly constrained, within a factor of two or less of the Simple Model.

- the remaining region, 'beyond' the steady states of the Linear Accretion Models:

$$p(1 - f) > Z$$

$$p(1 - f) < \langle Z\rangle$$

$$Zp_s/p < Z_{\rm S}/Z$$

can be reached by metal-free accretion through evolution, but only if the accretion rate inreases rapidly to a value in excess of the current star formation rate (i.e. $a > 1$), preferably in systems that are already chemically highly evolved (low gas fraction, high metallicity). Thus, a metallicity low in the gas but high in the stars and a low secondary-primary ratio could well be taken as evidence for a strong late accretion of gas by a galaxy.

To enable the reader to view the evolution of models with any arbitrary accretion one might wish to input, an interactive JAVA applet is available on the Internet at:
`http://astro.u-strasbg.fr/~koppen/ChemFlows.html`
`http://www.astrophysik.uni-kiel.de/pershome/supas086/ChemFlows.html`

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