Astrophysics with the Computer: Deflection of Cosmic Rays by the Magnetic Field of the Earth

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1 Astrophysics

Cosmic Rays are energetic particles (protons, electrons, α-particles, and other, heavier nuclei, which were discovered by observations on high-altitude balloons, but are present in interplanetary space and are an important component of the interstellar medium. The exact origin is still not completely known, but one thinks that they come from supernova explosions. This mystery exists partly because these particles are charged and therefore suffer deflection from the magnetic fields in interstellar space, in the heliosphere of our Solar System, and also from the Earth's magnetic field.

At any rate, we on Earth can call ourselves lucky that our planet has a magnetic field, because it shields us from this rather harmful radiation which would cause genetic mutations. we have become aware of this especially because of the human flight into space, and the necessity to protect humans against these harmful particles, which are also present in the Van Allen radiation belts and during solar eruptions.

In the 1930s the Norvegian geophysicist Carl Størmer worked out how these charged particles are deflected by the Earth's magnetic field, which is essentially the field of a magnetic dipole. He could show how and why not all particles reach the Earth surface, and that only the most energetic ones are able to do this. Also, he showed that this shielding is most effective near the geomagnetic equator while at the poles the particles are more able to penerate to the Earth, which is also the reason why near the poles one observes the phenomenon of the aurorae.

In this exercise we shall treat the problem in a simplified way, by neglecting relativistic effects. But this will still allow to see the essential results of Størmer's studies.

2 The equations

The Earth magnetic field can be approximated by the field of a dipole, thus the components of the magnetic field (in Cartesian coordinates x, y, z) are:

$$B_x = -M\frac{3xz}{r^5} \quad (1)$$

$$B_y = -M\frac{3yz}{r^5} \quad (2)$$

$$B_z = -M\frac{3z^2 - r^2}{r^5} \quad (3)$$

with the magnetic moment M, and $r = \sqrt{x^2 + y^2 + z^2}$.

The value for the magnetic moment M can be determined from the measurement of the magnetic field in central Europe of about 0.4 Gauss for the total value. Note that the SI unit 1 Tesla for B corresponds to 10^4 Gauss.

A particle of charge q, having a speed v, is subject to the Lorentz force:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{4}$$

This force will cause an acceleration of the particle (having a mass m):

$$m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \mathbf{F} \tag{5}$$

Here we neglect the relativistic effect that the mass depends on the velocity. The velocity is nothing but the rate of change of the position:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \tag{6}$$

3 How to solve the equations

We may use a simple explicit method to compute from the current forces and accelerations the velocities, here written for the x-coordinate only

$$v_x(t + \Delta t) = v_x(t) + a_x(t) \times \Delta t \tag{7}$$

Hereby we assume that the acceleration, computed from the current position of the particle, remains constant during the small, but finite time step Δt. Thus we obtain new values for the velocity components.

This integration over the time step could also be done in a more accurate way, by using the Runge-Kutta method, or any other method you like to try!

From the velocities, we can compute the new positions of the particle: We could either use the 'old' velocities:

$$x(t + \Delta t) = x(t) + v_x(t) \times \Delta t \tag{8}$$

or the 'new' ones:

$$x(t + \Delta t) = x(t) + v_x(t + \Delta t) \times \Delta t \tag{9}$$

The first method would be called completely 'explicit', because one uses only information available at the 'old' time instant. The second method would be called 'implicit' or rather 'semi-implicit', as it uses partly information from the new instant. If one chooses a sufficiently small time step, the difference between the two approaches become negligibly small. Furthermore, the equations are not nasty and present no fundamental problems, so that the choice of the method is not a crucial issue.

The calculation then proceeds as follows: at the new position, we compute the new forces, accelerations, and then go through the same steps again. In this way, we compute the evolution of the position and velocity of the particle, starting from the initial condition, i.e. the initial position and the initial velocity components. This means we have to specify six initial values!

4 Initial values

The basic procedure is to start the particle at some place far from the Earth, specify its initial velocity, and then integrate the equations of motion, until the particle either hits the Earth surface or it becomes clear that it will not hit the Earth.

Our problem is axisymmetric with respect to the polar axis of the magnetic field. Thus the azimuth of the initial position is without significance. We might choose to start always from the positive x-axis and 'shoot' the particle towards the Earth. What will matter is whether we shoot the particle directly towards the centre of the Earth or rather aim a bit 'to the left' or 'right', so to speak ... Does it matter whether it's left or right? Similar considerations are required for the position above and below the equatorial plane.

5 Time Step

The simplest thing is to use a constant time step ... and simply try out a good value which gives results fast but with sufficient accuracy. When in doubt, repeat the computation with a smaller time step!

It might be that under certain conditions the evolution needs a large time step during one phase of the flight, but a much smaller one during another phase. If this occurs, it would be better to install an automatic control of the time step; for instance one could demand that the change in velocity and position should always be smaller than certain values δ and Δ:

$$\Delta > (|v_x| + |v_y| + |v_z|) \times \Delta t \tag{10}$$

hence

$$\Delta_x t = \frac{\Delta}{(|v_x| + |v_y| + |v_z|)} \tag{11}$$

and in the same way from the accelerations

$$\Delta_v t = \frac{\delta}{(|a_x| + |a_y| + |a_z|)} \tag{12}$$

So that one could take

$$\Delta t = \min(\Delta_x t, \Delta_v t) \tag{13}$$

Or something like that ...

6 Suggestions for testing

Our problem is axisymmetric with respect to the polar axis. Therefore we should get the same type of solutions if we start from points at the same distance but at different geomagnetic longitudes.

Because of the magnetic polarity, there is no mirror symmetry with respect to the geomagnetic equatorial plane, but there is point symmetry with respect to the Earth centre. So trajectories from particles coming from the North would be point symmetric to trajectories from the South.

Of course, positively charged particles would behave like negatively charged ones going in the opposite direction.

7 Suggestions of how to proceed

First of all, one should verify that the code integrates the equations with sufficient accuracy, and that the integration method really works the way it should.

Then, one should try out various initial parameters to study what sort of behaviour the solutions show. Systematic studies of each parameter will help to find the interesting aspects. Also, these investigations should permit to ensure that the method works well under all conditions.

If that is the case, one could make automatic scans of the parameter space and produce maps of the regions on the Earth where particles from a certain direction and with cetrain energy will reach the surface. Or, make a map of the abstract space of energy and latitude from where particles can hit the Earth. This would give as a function of geographic latitude the minimum energy of particles reaching the surface ...

One can do systematic scans in parameters, such as:

- place the particle at the same distance from the Earth, but at various directions, and shoot the particle towards the Earth
- start at some initial position, and with the same speed, but vary the angle of the velocity vector

The variation of the parameter(s) could be done on a equidistant grid.

Another very nice posssibility is to do a Monte Carlo simulation: Here we pick the random parameter values for the next simulation. The result of each simulation is represented by a symbol or colour-coded dot on a plot of the parameter space. For instance, one could mark those areas in the initial direction of the sky whose particles arrive on the northern hemisphere. By executing many simulations, one can build up a nice map of this parameter space. Evidently, we must be sure that the program works correctly, rapidly, and without getting stuck somewhere! In such an approach we should also take care that the parameter values are evenly distributed ... For example if one varies the direction of the velocity vector in its two dimensions (θ, φ) by simply picking uniformly distributed random values of θ from the interval 0 ... π and of φ from 0 ... 2π, the directions will be concentrated to the poles. A truely isotropic distribution can be achieved, if we pick φ from a uniform distribution in 0 ... 2π, but $\cos\theta$ uniformly from -1 ... 1. You find more information about making random numbers distributed arbitrarily in Numerical Recipes, under the Transformation Methods for Random Numbers.

In Størmer's papers descriptions of his numerical methods and approximations dominate, because he had to use mechanical calculating machines ... but it should be possible to reproduce some of his plots and maps, but since not all his papers are available on the ADS and thus we have not all details for his plots, we should be content to reproduce the same behaviour of his results.

8 Literature

Articles by Carl Størmer 1931 ... 1937, available from ADS. Unfortunately not all articles are available, and not all are written in English!

The book "Numerical Recipes" is available in the MS2 library.

There is a test version of a Java applet - but still without any explanations – available at
`http://astro.u-strasbg.fr/~koppen/cosmicray/`