Astrophysics with the Computer: Internal Structure of Stars

Joachim Köppen Strasbourg 2002/03

1 Warning

This text may contain errors.

2 Astrophysics

Stars are probably the most important things in the universe. They are self-gravitating balls of hot gas, heated by internal nuclear fusion processes, which makes them shine and supply radiation to the surroundings, but which are also responsible for converting primordial elements (hydrogen and helium) into heavier elements ("metals": C, N, O, Fe, etc. which are vital ingredients of life). Stars are formed out of interstellar gas by gravitational instabilities, they live between a few million years to longer than the age of the universe in a nearly static equilibrium, and finally explode or fade away. Stars are the major constituents of galaxies, and their movement in space determines the shape of galaxies.

The physics of stars is very well known, which permits us to reliably compute not only the spectrum of a star, but also its internal structure which we cannot observe directly. With the knowledge of nuclear reaction rates measured in laboratories, we can quite well compute the evolution of stars. In this exercise, we shall tackle how a main sequence star looks inside, based on the basic physical processes.

3 Simple Model by Lane and Emden

We shall deal with the equilibrium structure of a non-rotating main–sequence star only, thus avoiding the difficulties of the dynamics of exploding stars, the extended atmospheres of giant and supergiant stars, and the deformations of rapidly rotating stars. Our prescriptions may also be used to describe white dwarfs, before they eventually crystallize.

Because of spherical symmetry, we need to consider only the equations for the radial variations of quantities, i.e. the problem of the structure of a 3-D object reduces to one dimension. The first equation is that of hydrostatic equilibrium: The pressure gradient across a gas element is balanced by its weight

$$\frac{dp}{dr} = -g\rho \tag{1}$$

where g is local gravitational acceleration. Because of spherical symmetry, it is simply due to the attraction of the mass inside a sphere of radius r:

$$g(r) = \frac{G}{r^2} \int_0^r 4\pi\xi^2 \rho(\xi) d\xi = \frac{Gm}{r^2} \tag{2}$$

For convenience we shall introduce the mass m up to radius r; this is often also called M_r. The pressure of the gas is related to density and temperature via the equation of state: We shall assume either an ideal gas

$$p = \frac{k_\mathrm{B}}{m_g} \rho T \tag{3}$$

(with m_g the mean mass of a gas atom) or a polytropic relation

$$p = K\rho^{(1+\frac{1}{n})} \tag{4}$$

which includes $n = 0 \rightarrow \rho = const.$ The polytrope is useful, as e.g. a completely convective star can be well described by a $n = 3/2$ law.

We may write the equation for m as a differential equation:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{5}$$

If we specify the equation of state, and the temperature, Equations 1 and 5 form a system of differential equations which describe the distribution of mass (and pressure) in an isothermal star. This does not seem too realistic, but it is not too bad for a rough first guess! What do we know about the boundary conditions? The mass m (within radius r) is of course zero in the centre: $m(0) = 0$. The pressure we do not know in the centre, but at the surface it is small, so we may demand $p(R) = 0$. The stellar radius R is defined as the radius where the mass becomes equal to the prescribed stellar mass $m(R) = M$.

So we have a system of two differential equations with conditions for the variables that are specified at the centre and at the surface. If both were specified say at the centre, we could simply integrate the equations, starting at the centre..... There are two methods to solve this more tricky problem:

The **shooting method** is the simplest. We guess some value $p(0)$ for the unknown central pressure, and we simply integrate both equations up to the surface, where we test whether the outer boundary condition is fulfilled. By changing the guess value, we can find that solution which does the job!

The other method is the **relaxation method** which starts with an initial guess for the solution, and one iteratively improves them until they both meet the boundary conditions. This method is more complicated, but more powerful, and is described in the Section on Numerics.

Defining $z = Ar$ with $A^2 = 4\pi G\rho_c^{1-1/n}/((n+1)K)$ and $w = (\rho/\rho_c)^{1/n}$, this problem leads to the Lane–Emden equation, whose solutions were computed about 100 years ago. In fact, there are a three analytical solutions

$$n = 0: \qquad w(z) = 1 - z^2/6 \tag{6}$$

$$n = 1: \qquad w(z) = \frac{\sin z}{z} \tag{7}$$

$$n = 5: \qquad w(z) = (1 + z^2/3)^{-1/2} \tag{8}$$

For the other values, one has to do it numerically. The table gives the value of z at the star's surface (where $w = 0$) and the ratio between central density and mean density.

n	$z_{\rm surface}$	$\rho_c/\bar{\rho}$
0	2.4494	1.0
1	3.14159	3.28987
1.5	3.65375	5.99071
2	4.35287	11.40254
3	6.89685	54.1825
4	14.97155	622.408
4.5	31.8365	6189.47
5	∞	—

3.1 A Model for the Sun

To make a simple (isothermal or polytropic) model for the internal structure of the Sun, we shall demand that the model should be such that it should both reproduce the radius and the mass of the Sun, which can be measured.

Let us take the case of the isothermal model first: The unknown free parameters of the model are the temperature and central density. Our task is to determine both parameters by simultaneously matching the constraint $m(R_\odot) = M_\odot$. This can be done as shown in the Figure: the plot on the right hand side shows the tracks in the $r - m$ plane of several models whose temperatures and central densities are marked in the plot on the left hand side. All model tracks start at the centre with $r = 0$ and $m = 0$, but depending on their parameters they either meet the $r = R_\odot$ condition before or after the $m = M_\odot$ condition. Making several tries we mark in the $T - \rho_0$ plot the models according to their behaviour, which will form two different regions in this plane. Our aim is to find the model which meets $r = R_\odot$ and $m = M_\odot$ at the same radial shell. Evidently, these models are found on the border between the two regions. This evidently is a curve, so we have an infinite varfiety of possible models. Now we make use of the outer boundary condition, that the pressure at the surface should be much smaller than in the centre – or one could in principle demand that the pressure should be a certain value.

The same technique we can apply to find the polytropic model (with a given polytropic index): the free parameters now are the polytropic constant K and the central pressure (or density).

3.2 Stars with Energy Production

Let us now make the star more realistic, by taking into account that there are internal sources of energy, that the energy is transported by radiation to the outside, and that the interaction of the gas with the radiation determines the gas temperature.

The production of energy increases the luminosity L_r that the star has inside a radius r:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon \tag{9}$$

where ϵ is the energy production rate per unit mass of the gas. In a main sequence star, the dominant nuclear fusion process is the burning of hydrogen by the pp–chains and the CNO–cycle. From Kippenhahn and Weigert's book, we get the prduction rates for the pp–chain

$$\epsilon_{\rm pp} = 2.38\,10^6\ \Psi f_{11} g_{11} \rho X_1^2 T_6^{-2/3} \exp(-33.80/T_6^{1/3}) \tag{10}$$

$$g_{11} = 1 + 0.0123 T_6^{1/3} + 0.0109 T_6^{2/3} + 0.0009 T_6 \tag{11}$$

with $T_6 = T/10^6$ K, $f_{11} \approx 1$, $1 < \Psi < 2$ (see Fig. 18.7), and X_1 the mass fraction for hydrogen (≈ 0.75) in the gas. The rate for the CNO-cycle

$$\epsilon_{\rm CNO} = 8.67\,10^{27}\ g_{14,1} \rho X_1 X_{\rm CNO} T_6^{-2/3} \exp(-152.28/T_6^{1/3}) \tag{12}$$

$$g_{14,1} = 1 + 0.0027 T_6^{1/3} - 0.00778 T_6^{2/3} - 0.000149 T_6 \tag{13}$$

depends on the mass fraction of CNO–elements $X_{\rm CNO} \approx 0.02$. If you need perhaps in more massive stars, here is the rate for Helium burning due to the triple-α–process:

$$\epsilon_{3\alpha} = 5.09\,10^{11}\ f_{3\alpha} \rho^2 X_4^3 T_8^{-3} \exp(-44.027/T_8) \tag{14}$$

The energy production rate has to be modified because due to the escape of neutrinos, some energy is lost. Kippenhahn and Weigert give the appropriate expressions.

We shall assume that the transport of energy is only due to radiation (so we neglect convection, which would not be too complicated to take into account). With the diffusion approximation of the radiative transfer (which is o.k. in the interior of the star, but is a poor approximation near the surface where the visible spectrum is formed) we can compute the resulting temperature gradient:

$$\frac{dT}{dr} = -\frac{3\rho\kappa L_r}{16\pi a c r^2 T^3} \tag{15}$$

where $a = 7.57\,10^{-15}$ erg cm^{-3} K^{-4} is the radiation–density constant. The Rosseland mean opacity κ is the opacity of the gas, averaged over the spectrum, and has been computed based on atomic cross sections, the equation of state, ionization and excitation of all elements and ions, and for a certain chemical composition of the gas. It depends on density and temperature. Usually it is given in the form of data tables (Cox & Stewart (1965, 1970), Meyer-Hofmeister (1982), Carson (1976), Alexander et al. (1983)) In the

book by Kippenhahn and Weigert there is a nice plot (Fig. 17.6) from which one may cook up a simple rough analytical prescription.... It is convenient to define the **radiative gradient**

$$\nabla_{\rm rad} = \left(\frac{d\ln T}{d\ln p}\right)_{\rm rad} = \frac{3\kappa L_r p}{16\pi acGmT^4} \tag{16}$$

With these two further differential equations we can completely describe the interior of the star. Here are the four equations:

$$\frac{dm}{dr} = 4\pi r^2\rho \tag{17}$$
$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \tag{18}$$
$$\frac{dL_r}{dr} = 4\pi r^2\rho\epsilon \tag{19}$$
$$\frac{dT}{dr} = -\frac{3\rho\kappa L_r}{16\pi acr^2T^3} \tag{20}$$

The boundary conditions are, respectively

$$m(0) = 0 \tag{21}$$
$$p(R) \approx 0 \tag{22}$$
$$L_r(0) = 0 \tag{23}$$
$$T(R) \approx 0 \tag{24}$$
$$\tag{25}$$

where R is the radius of the star we want to model. The mass of the star is $m(R)$ and its luminosity $L_r(R)$. The surface conditions are indicated as approximative, because both p and T are not really zero, but have values much smaller than in the interior. We shall set them to zero, but a proper stellar structure and evolution code would use some prescriptions of how the interior is linked to the stellar atmosphere.

When we use the shooting method, we now have to guess **two** values at the centre, namely the central density ρ_0 (or pressure p_0) and the temperature T_0. And we have to look for **two** conditions to be met at the surface, i.e. that pressure (or density) and temperature become nearly zero. This makes the search more complicated, but we can adapt our technique which we used in the isothermal model to determine the two free parameters T, ρ_0 from the constraint $m(r = R_\odot) = M_\odot$. We plot tracks in the pressure-temperature diagram of models with different central values p_0, T_0 and note which models first meet the $r = R_\odot$ or $m = M_\odot$ conditions. Again, one obtains several solutions, among which one selects those whose temperature and pressure at the surface are sufficientlty small.

In this way we can compute the structure of stars of a given radius. It would be much nicer to be able to specify the mass of the star rather than getting it as a result of the computations. However, writing the structure equations with the shall mass m as independent variable leads to very nasty numerical problems, which can only be treated in a

rather complicated way. This is why we shall use here the radial coordinate.

Another problem is the fact that the density and temperature change strongly near the surface. This can be dealt with by either using an adaptive radial grid whose spacing is automatically made smaller when strong gradients in ρ and T appear. Or we simply use a fixed radial grid whose spacing is made finer towards the surface.

Historically, the shooting method was used in the first stellar structure calculations, but it is inefficient and becomes complicated when one deals with more than simplified models. A much better method is solving the system of equations by the relaxation method.

Stellar evolution: The computation of how a star evolves in time is very similar to the structure code. Given the structure of the star at some time, one computes from the reaction rates the change of the chemical composition (as hydrogen is comsumed and helium and other elements are synthesized) during a given time interval. Then one computes the structure of the star with this changed composition profile, and so on.... it is just a sequence of equilibrium models. Of course, phases of dynamic evolution, such as supernova explosions, have to be done differently, with a really dynamic code.

4 Numerical Methods

4.1 Integration of Initial Value Problem

To integrate the equation $dy/dx = f(x, y)$ starting from the initial condition $y(x_0) = y_0$, we may employ a simple method, such as to apply for all $i = 1, ..., n$

$$y_i = y_{i-1} + f(x_{i-1}, y_{i-1})\Delta x \tag{26}$$

with some suitably small step Δx which is here in radial direction.

You may also use more sophisticated methods, such as Runge-Kutta, which gives a more accurate solution for the same size of the stepwidth.

4.2 Relaxation Method

Let us derive this method for the same problem as before: Solve $dy/dx = f(x, y)$ starting from $y(x_0) = y_0$. The formal solution is

$$y(x) = y_0 + \int_{x_0}^{x} f(x', y(x'))dx' \tag{27}$$

Let us discretise the interval $x_0...x_n$ into x_i for $i = 1, ..., n$. This does not need to be done in steps of equal size, but we need to know the weights w_i to evaluate the integral numerically:

$$y_i = y_0 + \sum_{j=0}^{i} f(x_j, y_j)w_j \tag{28}$$

(with $y_i = y(x_i)$) which is the numerical equivalent of Eqn. 27. Now, a given numerical solution y_i for $i = 1, ..., n$ is a solution of our problem, if Eqn. 28 valid for **all** indices i.

For the simple trapezoidal method of integration, the weights are simply:

$$w_a = (x_{a+1} - x_a)/2 \tag{29}$$
$$w_i = (x_{i+1} - x_{x-1})/2 \tag{30}$$
$$w_b = (x_b - x_{b-1})/2 \tag{31}$$

for an integration between x_a and x_b. But note that in the sums of Eqn. 28 one has to keep track that the upper limit of integration is not fixed!!!

(There is an alternative formulation, based on the direct discretization of the differential equation $dy/dx = f(x, y)$, for example

$$\frac{dy}{dx} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \tag{32}$$

which from an explicit formulation gives the recursion formula

$$y_i = y_{i-1} + (x_i - x_{i-1}) \cdot f(x_{i-1}, y_{i-1}) \tag{33}$$

which we could use instead of Eq. 28.)

In any case, if we could only solve this system of simultaneous equations for the y_i, we could get the solution directly! If the function $f(x, y)$ were linear in y, we would deal with a system of n linear equations for the n unknown y_i, and is could easily be done by Gauss elimination!

However, in general, f will not be linear in y. So we have to design an iterative method to solve for the y_i: Let us expand f into a Taylor series, and keep only the linear terms:

$$f(x, y + \Delta y) \approx f(x, y) + \frac{\partial f}{\partial y}\Delta y \tag{34}$$

and let us put that into Eqn. 28:

$$y_i = y_0 + \sum_{j=0}^{i} f(x_j, y_j) w_j \tag{35}$$
$$+ \sum_{j=0}^{i} \frac{\partial f(x_j, y_j)}{\partial y} w_j \cdot \Delta y_j \tag{36}$$

This is a system of **linear** equations for the Δy_i, which can be solved easily! So, given a guess solution, i.e. a set of y_i, we solve the above system to get corrections for the y–values, which give us the new guess solution. This process is repeated until the corrections are sufficiently small, say $\max_i |\Delta y_i| < \epsilon$.

The calculation of the derivatives $\partial f/\partial y$ can be done analytically (if the expressions are not too complicated) or numerically. For the solution of the system of linear equations, the Gauss elimination can be used, if the matrix involved is not too large. Otherwise, the rounding errors caused in the many additions may completely destroy the solution; this depends on the structure of the matrix. If you run into troubles here, one may use one of the FORTRAN routines of the NAG library, or take routines from "Numerical Recipes" by W.H.Press et al. (disponible de la bibliothèque du DEA).

Before you try to apply a relaxation method to the full problem, develop a program to solve a simple equation, for example $f(x, y) = -y$ for which you know the exact solution, and for which you can easily work out the coefficients of the matrix and all intermediate points y_i of the numerical solution. You may notice that your numerical solution will not be the exact solution, and we may need to develop some more tricks!

In this iteration process, you should carefully monitor the behaviour: The corrections should always decrease in size! If they are always in the same direction but one needs many iterations to get to a decent accuracy, one might try to use corrections somewhat larger than given by the method. But be careful not to overdo it! On the other hand, if the corrections always change sign, i.e. perform an oscillation – hopefully a damped one – it might be a good idea to use corrections smaller than given by the method.....

The structure of the matrix of coefficients for the Δy_i depends on which side(s) the boundary condition is given. What will the shape be for our simple initial-value problem?

This method is easily extended for several equations, say k, with the boundary conditions of each equation on either side. The coefficient matrix then may look more complicated, but it gives the corrections for the each of the n values for each of the k variables. The matrix will of course be k times larger than with one equation. If you have problems with solving this large system, one might break up the problem into solving the k sub–systems one after the other. In each case, one takes all other variables as constant. This sequence of solutions is iterated until all results are sufficiently stable.

5 Tasks

- solve Eqn. 1 and 5 with the shooting method. Use the isothermal case, and start integration at the centre If you take a value of $T = 140\,000\,000$ K for the interior of the Sun, what pressure (and density) would be at the centre?
- Again, solve Eqn. 1 and 5 with the shooting method, but for polytropic models, and verify the results from the Lane–Emden models
- construct models for the Sun, both isothermal and polytroipic ones, using the 2-D search scheme
- Apply the relaxation method. Verify the results from the Lane–Emden models

- solve all four equations with the shooting method and the 2-D search. Compute the temperature at the centre of the Sun. Try stars with other masses!

6 Literature

The book: Kippenhahn and Weigert: *Stellar Structure and Evolution* (disponible à la bibliothèque du DEA) is a good textbook for this subject.

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