Astrophysics with the Computer: Evolution of Viscous Disks

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1 Warning

This exercise is still experimental. Your work and your experiences will help to better define this exercise. Your spotting mistakes and any comments are welcome.

2 Astrophysics

Accretion – the loss of potential energy of matter as it plunges down the gravitational well of compact objects – is a very efficient mechanism to convert gravitational energy into radiation. It is responsible for a wide range of high energy phenomena in space: Quasars are powerful emitters of radiation, being often much brighter than their host galaxy, but all radiation comes from a small region in the galaxy's centre, where a massive black hole breaks up stars by tidal forces and accretes the gas. On a much smaller scale, there are X-ray binaries which contain gas hot enough to make them bright in the X-ray range. Dwarf novae have periodic and stochastic outbursts (enhanced luminosity in optical and X-rays). Here, gas from a companion red giant star is accreted by white dwarfs, neutron stars, or small black holes.

Because of the conservation of angular momentum, a parcel of gas cannot reach directly a compact object, such as a star, by simply falling towards it radially. Any small amount of initial angular momentum would make the parcel to follow a hyperbolic orbit around the star. Thus radial accretion is very inefficient. On the other hand, a viscous disk provides a means for a chunk of matter to get rid of angular momentum, as the friction with other gas elements will convert kinetic energy into thermal energy which is radiated away into space. Thus the gas element constantly loses speed and performs a spiral orbit towards the central object.

The standard model for X-ray binaries and novae – Roche-lobe overflow into an accretion disk – explains the nova outbursts by an instability of the accretion disk switching from an low-temperature state into a high-temperature state because the vertical structure of the disk changes from opaque to transparent as gas gets ionized.

The concept of viscous disks is also important in other contexts: the gaseous disks in spiral galaxies are modeled ain this way, the viscosity being due to turbulence in the interstellar medium. Stars may also accrete matter by this process during their formation, forming planetary systems around them.

3 The equations

We represent the disk by a **thin** disk, i.e. we shall not consider the variation of physical quantities with height above the plane. Integrating over all heights, we no longer consider (volume) densities but surface (or column) densities. And all other properties are represented by their value projected into the plane. The basic equations are those of hydrodynamics, written in cylindrical symmetry, and integrated over all heights. The conservation of mass becomes

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial r v_r \Sigma}{\partial r} = S(r,t) \tag{1}$$

where S is the mass accretion rate – it would be zero everywhere except at the outer boundary, where the gas from the companion arrives. The radial drift velocity v_r is computed from the conservation of angular momentum $r v_\phi \Sigma$

$$\frac{\partial r v_\phi \Sigma}{\partial t} + \frac{1}{r}\frac{\partial r v_r r v_\phi \Sigma}{\partial r} = -\frac{1}{r}\frac{\partial r^2 \tilde{\tau}_{r\phi}}{\partial r} \tag{2}$$

with the azimuthal velocity v_ϕ. The term on the right hand side is the torque acting on the gas element, due to the viscosity. The r, ϕ component of the height-averaged viscous stress tensor $\tilde{\tau}$

$$\tilde{\tau}_{r\phi} = -r\nu\Sigma\frac{\partial \Omega}{\partial r} \tag{3}$$

depends on the viscosity ν and the angular frequency $\Omega = v_\phi/r$ of the disk's rotation. Often one considers a simple Kepler's law, if the mass of the disk can be neglected against the mass of the central object. Any part of the disk at radius r is assumed to be in circular orbit around the central mass M and

$$v_\phi(r) = \sqrt{\frac{GM}{r}} \tag{4}$$

If we can assume that the disk rotation is independent of time, we can simplify Eqn. 2 by taking out the derivatives of Σ. Then noting that for all points in the disk $S = 0$, we can subtract Eqn. 1 and finally get

$$\frac{\partial r^2\Omega}{\partial t}\Sigma + \frac{1}{r}\frac{\partial r^2 \Omega}{\partial r}\Sigma r v_r = \frac{1}{r}\frac{\partial}{\partial r}(r^3\nu\Sigma\frac{\partial \Omega}{\partial r}) \tag{5}$$

As Ω was assumed to be constant in time, the first term vanishes and one simply gets an equation for the drift velocity

$$v_r = \frac{\partial r^3 \nu \Sigma (d\Omega/dr)/\partial r}{\Sigma r (d(r^2\Omega)/dr} \tag{6}$$

Check whether for the keplerian case you get Eqn.(1) and the expression for v_r in Cannizzo (1993).

3.1 The basic viscous disk equation

For a kepler-disk we take the equation for the radial drift velocity (Eqn. 6) and put it into the momentum equation (2) to get

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left(r^{1/2}\frac{\partial r^{1/2}\nu\Sigma}{\partial r}\right) \tag{7}$$

e.g. Cannizzo (1993).

3.2 Stationary Solution

This standard equation has a stationary solution: setting time derivative to zero,

$$r^{1/2}\frac{dr^{1/2}\nu\Sigma}{dr} = const. \tag{8}$$

which means that the surface density profile would follow a law like

$$\nu(r)\Sigma(r) = const. \tag{9}$$

If we follow Cannizzo for the viscosity $\nu = (2/3)(\alpha RT/(\mu\Omega))$ then we get for a kepler-disk $\Sigma \propto r^{-3/2}$.

This solution is useful for checking the code:

- starting with such a profile, and taking a constant mass accretion rate, the code must stick to this solution. Any deviations may indicate some error in the program, and those which perhaps grow in time must be of numerical nature.
- starting with a rather different density profile, the program should approach the stationary case, and finally stay there
- changing parameters, such as α or taking different rotation laws, should result in the programm to find the other stationary solution

If not, you've got a problem!

4 Energy Balance

Next, we add the equation for the energy balance in order to compute the temperature T in the mid-plane. We take it from Cannizzo (1993):

$$c_p\Sigma\frac{\partial T}{\partial t} = 2(H - C + J) - \frac{R}{\mu}\frac{\Sigma T}{r}\frac{\partial(rv_r)}{\partial r} - c_p\Sigma v_r\frac{\partial T}{\partial r} \tag{10}$$

with the rate of viscous heating

$$H = \frac{9}{8}\nu\Omega^2\Sigma \tag{11}$$

the cooling of the gas by radiation

$$C = \sigma T_{\text{eff}}^4 \tag{12}$$

and the radial energy flux carried by viscous processes

$$J = \frac{3}{2} \frac{\nu \Sigma}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \tag{13}$$

The fourth term is the energy release from $p\Delta V$ work, and the final term is the advective transport of energy carried by flows over temperature gradients.

The effective temperature represents the radiation emitted by all the vertical layers of a section of the disk. Its computation requires solving the vertical structure of the disk, along with radiative transport and the equation of state of the gas (and ionization) all of which is rather complicated. For the thin disk approximation, one solves the vertical structure separately, and fits the results in some convenient forms.

Along with a variable temperature, we now must take ito account that the viscosity depends on temperature:

$$\nu = \frac{2}{3} \frac{\alpha R T}{\mu \Omega} \tag{14}$$

where α is a fudge-factor, which one adjusts to observations. People find that is is in the range of 0.0001...1. We shall keep this factor undetermined, and check our method if it works for any value of α!

4.1 Approximation $T_{\text{eff}} = T$

As a first step we shall make the crude approximation that the effective temperature is equal to the temperature of the gas in mid-plane.

For simplicity, we may first use for the specific heat a constant value:

$$c_p/R \approx 10 \tag{15}$$

and later the analytic expression (5) of Cannizzo (1993), to take into account that the gas becomes ionized above temperatures of 10000 K.

From the form of the energy equation, it is obvious that there must be a steady-state solution. Setting $\partial T/\partial t = 0$ gives an ordinary differential equation for the temperature profile $T(r)$. This can be easily done numerically. I haven't checked whether one could also do it analytically. Looks nasty, but perhaps you can try? Note that this steady state involves both the equation for σ and for T.

Of course one can also find this steady state soltuion but following the time evolution of the whole disk, starting from some arbitrary initial solution and settling into the steady state.

4.2 The S-curve (simplified)

Viscous disks have an interesting behaviour which is used to explain the periodic outbursts of cataclysmic variables: the disk may perform nonlinear oscillations between a state in which the disk is essentially cool, dense, and thin and a state where it is hot, less dense, and less thin.

If one computes the vertical structure in all necessary detail, such as radiative transfer, ionization of the elements, selfgravity, and using proper atomic data (opacity κ), one finds the true relations between Σ and the mid-plane temperature T and the effective temperature $T_{\rm eff}$. These are given in Cannizzo (1993).

Both at low and high temperatures $\Sigma(T)$ increases with T. But at about $T \approx 10000$ K, hydrogen (which is 90 percent of the gas) becomes ionized, and the opacity κ in the optical is substantially smaller, causing Σ to decrease with T. The plot of e.g. $T_{\rm eff}$ as a function of Σ thus shows the form of an "S". The portions with positive slope are stable, and there is a cool and a hot branch; in the middle section the vertical structure is unstable, as it would either heat up until it reaches the hot branch, or cool down.

Cannizzo gives for the hot branch

$$T_{\rm eff} = 3334{\rm K}\ M_1^{9/56} r_{10}^{-27/56} \alpha^{2/7} \mu^{-15/56}\ \Sigma^{5/14} \tag{16}$$

where $r_{10} = r/10^{10}$ cm and the mass $M_1 = M/M_\odot$ of the central object in solar masses. For the hot ionized branch, one has $\mu \approx 0.6$ since electrons contribute to the number density, but almost nothing to the mass density (Neutral matter of solar composition has $\mu \approx 1.3$). This fit formula is valid for

$$\Sigma \geq \Sigma_{\rm min} = 8.25{\rm g\,cm^{-3}}\ r_{10}^{1.05} M_1^{-0.35} \alpha^{-0.8} \tag{17}$$

and $T_{\rm eff,min} = 7951{\rm K}\ (M_1/r_{10}^3)^{0.05}$. The cool branch

$$T_{\rm eff} = 946{\rm K}\ r_{10}^{-0.8875} \alpha^{0.645}\ \Sigma^{0.75} \tag{18}$$

is valid for

$$\Sigma \leq \Sigma_{\rm max} = 11.4{\rm g\,cm^{-3}}\ r_{10}^{1.05} M_1^{-0.35} \alpha^{-0.86} \tag{19}$$

and $T_{\rm eff,max} = 5870{\rm K}\ r_{10}^{-0.1}$. The instable branch is computed by logarithmic interpolation between the two points min and max.

Rather than using the full recipe, as in Cannizzo (1993), I suggest that we investigate which shape for the S-curve is neccessary to get an oscillating disk. Let us take an artificial $\Sigma(T_{\rm eff})$–relation, whose shape we deform from a monotonic relation to the S-curve. One way to do this, is to define the positions of the turning points for the S-curve, and find a simple analytical function which goes through the two points.

In this way we can find out how the behaviour of the disk solutions changes, if one adds a non–linearity of this kind. As long as the S-curve feature is weak, the solutions will resemble what we had computed before. At some degree of S-curve, we will find that the solutions are different. Will this be drastic, will this be a sharp transition of behaviour? I do not know. Let's find out.

People have found that the presence of the S-curve alone does not make oscillating disks. To make well–separated outbursts, one needs the parametre α to be larger on the hot branch than along the cool branch. Cannizzo's (1993) model for SS Cyg assumes $\alpha_{\mathrm{cold}} = 0.02$ and $\alpha_{\mathrm{hot}} = 0.1$. How critical is such a choice?........

5 Why and how the disk oscillates

The basic behaviour of a viscous disk is that it likes to settle into its stationary solution. If one increases the inflow of gas at the outer rim, a density enhancemnt will propagate from the exterior towards the centre, and redistribute the matter, so that the new stationary solution is approached.

Suppose that the disk is perturbed by putting up the density at some intervall in radius. The excess matter will flow towards the interior – for constant temperature models you can show that analytically by checking what the sign of the radial drift speed would be! Thus, the density at that particular intervall will decrease. Consider now what happens when we treat a disk with the energy equation. Let us assume that the disk in this intervall is in the hot state. As the density drops, so will the effective temperature, but this will occur only until we reach the end of the hot branch. At that point, the temperature drops suddenly to the cold branch. Since α on the cold branch is smaller, which means a lower viscosity, and hence a smaller radial drift velocity. This means that the matter flow from the outer regions will be slower, and because of the accumulation of gas in the intervall the density will increase. So we move up on the cold branch, until its end, where we jump up to the hot branch. The larger viscosity on the hot branch will make the radial streaming faster, the density decreases and so on ....

The existence of this cyclic behaviour (called Limit Cycle) requires that the densities are always in the proper range, and that the jump in α is the proper size. So, if the mass accretion rate is too high, the densities remain too high to make possible such behaviour. The disk would always remain in its hot state. Likewise, if the accretion rate is too low, the disk will remain in the cool state. The range of values for the accretion rate, which make the disk oscillate, may be small, may be large, may be at astrophysically realistic values – it depends on the shape of the S-curve, and the various other parameters .... When you have found a parameter set for an oscillating disk, change one parameter to see whether it is a critical parameter and how abruptly the oscillations stop....

6 How to solve it numerically

6.1 Finite Difference Method

We shall use the method of finite differences. This means that we divide space and time (here: radius and time) into many small intervals – the sizes of these intervals must be chosen so that one get a sufficient resolution. Thus, all the functions of the problem (here: density, temperature, velocity, etc.) are thus given only on these discrete points. All derivatives will be approximated by their differences on these grids. For example,

$$\frac{df}{dt} = \frac{f(t_2) - f(t_1)}{t2 - t1} \tag{20}$$

To solve the differential equation $dy/dt = f(y, t)$ we get a scheme to compute from the 'old' value $y(t)$ the 'new' value at one time-step later:

$$y(t + \Delta t) = y(t) + \Delta t \cdot f(y(t), t) \tag{21}$$

Note that we've used on the right-hand side only data that is known at the 'old' time, so we use an explicit method. The size of the time step Δt determines the accuracy of the calculation, obviously a smaller time-step makes it more accurate.

To solve a partial differential equation, we chose e.g. the simple explicit method above (which is a 1st order method) to do the time stepping. All other derivatives (in radius) are also computed from formulae of this type, but evaluated of course at the old time:

$$\frac{\partial}{\partial r} f(r_i, t) \approx \frac{f(r_i, t) - f(r_{i-1}, t)}{r_i - r_{i-1}} \tag{22}$$

This is a simple first order recipe, which might also cause problems, because it is asymmetrical in radial direction. The simple second-oder scheme is symmetrical

$$\frac{\partial}{\partial r} f(r_i, t) \approx \frac{f(r_{i+1}, t) - f(r_{i-1}, t)}{r_{i+1} - r_{i-1}} \tag{23}$$

but it has an instability, because the derivative at radius i does not depend on the function at that point....

6.2 How to make differencing formulae

In a finite difference method we need formulae that approximate differential quotients by difference quotients. Here is a reminder how to make such a formula with a given order of accuracy:

Suppose I want to approximate the derivative in the central point x_2 by using three points of the grid (x_1, x_2, x_3), and I demand second-order accuracy

$$\frac{dy}{dx} \approx A \cdot y(x_1) + B \cdot y(x_2) + C \cdot y(x_3) \tag{24}$$

First, we express the function $y(x)$ by its Taylor series about the (desired) midpoint (with $h = x - x_2$):

$$y(x) = y_2 + h \cdot y_2^{(1)} + \frac{h^2}{2} \cdot y_2^{(2)} + \frac{h^3}{3!} \cdot y_2^{(3)} + ... \tag{25}$$

or

$$dy/dx = y_2^{(1)} + \frac{h}{2} \cdot y_2^{(2)} + \frac{h^2}{3!} \cdot y_2^{(3)} + ... \tag{26}$$

When we demand second-order behaviour

$$dy/dx = y_2^{(1)} + O(h^2) \tag{27}$$

this means that the term depending on $y_2^{(2)}$ should vanish. On the other hand, we evaluate the right hand side of Eqn. 24 putting in the Taylor expansions

$$\begin{aligned} dy/dx &= A(y_2 + h_{12} \cdot y_2^{(1)} + \frac{h_{12}^2}{2} \cdot y_2^{(2)} + ...) \\ &+ By_2 \\ &+ C(y_2 + h_{32} \cdot y_2^{(1)} + \frac{h_{32}^2}{2} \cdot y_2^{(2)} + ...) \end{aligned} \tag{28}$$

with $h_{ik} = x_i - x_k$. If this should be equal to Eqn. 27 up to 2nd order for any function $y(x)$, we get a system of linear equations for the coefficients A, B, C:

$$\begin{aligned} A + B + C &= 0 \\ A \cdot h_{12} + C \cdot h_{32} &= 1 \\ A \cdot h_{12}^2/2 + C \cdot h_{32}^2/2 &= 0 \end{aligned} \tag{29}$$

which gives

$$\begin{aligned} A &= -\frac{h_{32}}{h_{12}(h_{12} - h_{32})} \\ B &= -\frac{h_{12}^2 - h_{32}^2}{h_{12}h_{32}(h_{12} - h_{32})} \\ C &= \frac{h_{12}}{h_{32}(h_{12} - h_{32})} \end{aligned} \tag{30}$$

This is the generalization for the well known formula for an equi-spaced grid ($h_{32} = -h_{12} = h$):

$$\begin{aligned} A &= -\frac{1}{2h} \\ B &= 0 \\ C &= \frac{1}{2h} \end{aligned} \tag{31}$$

With this method one can derive formulae for any derivative at any point with any accuracy. Obviously one has to choose the number of points in accordance with the required accuracy!

6.3 The grids

For the discretisation in time, we shall take the simplest one, i.e. with a constant time-step. What size you need, depends on the accuracy you want, and is best found out by experiments. It should not be too large, as the program will tell you rather quickly by 'exploding' sooner or later.

For the discretisation in space, let us first try a simple radial grid with constant interval. Since the problem has a symmetry centre at the origin, and terms with $1/r$ will grow there beyond any limit, it might be a good idea to concentrate the grid towards the origin. Bath & Pringle and others use a grid equally space in $\sqrt{r}$ rather than the simple linear one. This means that one cannot use discretizaion formulae built for constant step size, or one transforms the problem into that space, by working not with radii, but with $\sqrt{r}$. I recommend that you try out various things. Which one do you find to be the best one???

For a method stepping explicitly in time, the Courant-Friedrichs-Levy condition must be obeyed, if the method should not explode. The time step dt must be smaller than the time required for the *fastest* element to cross a radial grid cell:

$$dt < \min_i |\frac{\Delta r_i}{v_i}| \tag{32}$$

Usually, one does not take the full time step, permitted by this formula. See what happens if you don't obey it!

6.4 Boundary conditions

In our problem the matter arrives at the outer rim, and flows towards the inside. At the outer boundary we thus have to specify how much mass arrives in each time interval...

A simple method is to keep the surface density at the outermost point (r_n if we have n radial points) fixed to some value which depends on the mass accretion rate:

$$\dot{M} = 2\pi r_n \cdot \Sigma(r_n) \cdot v_r(r_n) \tag{33}$$

With the formula for the drift velocity (Eqn. 6) and the expressions for rotational frequency Ω and viscosity and the current density profile, one could in principle compute the required density $\Sigma(r_n)$. I haven't tried it, but it might be just some complicated non-linear equation which one could crunch interatively by the bi-section methode (French: "dichotomie"). May be you try it?

6.5 Transforming the equations

Often, it is not numerically wise to solve the equation in its original form. We are dealing with a problem with rotational symmetry. So the conservation of mass can also be written as

$$\frac{\partial r\Sigma}{\partial t} + \frac{\partial r v_r \Sigma}{\partial r} = rS(r,t) \tag{34}$$

So we might use $r\Sigma$ as a function instead of Σ. Since the surface density Σ will turn out to trend to large values as one goes towards the centre (cf. the stationary solution of the kepler-disk), the new variable peaks less at the centre, and the difference formulae may not make so large errors. In fact, one could perhaps use the profile of the stationary solution $r^{3/2}\Sigma = const.$ as a new variable!

6.6 Conservative Formulation

The way I've written the Eqns. 1 and 2

$$\frac{\partial A}{\partial t} + \frac{1}{r}\frac{\partial rvA}{\partial r} = S_A \tag{35}$$

emphasizes that A is a conserved quantity (in rotationally symmetric coordinates). The second term is called advection; it counts how much of A is transported from and into a volume by the velocity v. The term on the right is a source term, telling how much is locally produced (or destroyed) of A. By formulating a discretisation recipe which ensures the balance of the three terms, one can build a code which strictly conserves quantity A. Here one would need a scheme in which all that is computed to leave radial element i with a positive velocity is equal to what is computed to arrive at radial element $i+1$, its neighbour.

7 Tasks

Here are some suggestions how to proceed:

- make kepler-disks, solving only the equation of mass conservation together with the formula for the radial drift velocity. Keep Σ fixed for outermost point, and check for the stationary solution, and whether you can reach it from whatever you start with.
- experiment with the radial grid and its discretisation recipes
- add a time-dependent mass accretion, and see how the disk takes up this mass, distributes it in a new stationary solution. How much time does this take?
- next one adds the equation for the temperature, and finds the stationary solution for the approximation $T_{\rm eff} = T$
- add the S–curve and try to find oscillations. Does one get oscillations if one assumes the same α for both hot and cool branch? Does one get oscillations for all mass accretion rates?.....
- if you could formulate tasks that you find interesting, it would be nice!

8 Literature

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