Towards a Newtonian softening length in discs simulations?

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Abstract

In numerical simulations of discs, self-gravitating potentials and forces are traditionally computed from softened gravity (see **panel 1**). In this context, the softening length λ is set to a fraction of the disc local thickness h, typically $\lambda/h \in [0.3 - 1.2]$. As shown (see **panel 2**), such a prescription is not appropriate. We have computed the gravitational potential of a numerical cell, and deduced the corresponding λ-value. It turns out that the length not only depends on the shape of the cell but it can be an imaginary number (see **panels 3 and 4**). A dipolar expansion shows that λ effectively does not depend on the cell's height only (see **panel 5**). We present a novel prescription, valid at long-range, that preserves the Newtonian properties at the scale of the numerical grid cells. A general analysis is in progress.

1. The softened point mass potential

The softened potential is used in continuous media to simplify the numerical treatment of Newton's triple integral. It avoids the kernel singularity through a modification of the relative separation, namely

$$|\vec{r} - \vec{r}'| \to \sqrt{|\vec{r} - \vec{r}'|^2 + \lambda^2}, \quad (1)$$

where λ is called the "softening length". This free parameter is selected by authors more or less arbitrarily [Papaloizou and Lin, 1989, Morishima and Saio, 1994, Baruteau and Masset, 2008, Meru and Bate, 2012]. In a discretized disc, the potential of each cell is replaced by the softened point mass potential (also known as Plummer potential), namely

$$\psi_{\rm Plum.}(\vec{r}; \vec{r}_0', \lambda) = -\frac{Gm}{\sqrt{|\vec{r} - \vec{r}_0'|^2 + \lambda^2}}. \quad (2)$$

The value of λ has a severe impact on the simulations. The figure 1 shows the error on the gravitational potential as function of the radius and ratio λ/h in two concrete cases. We see that the nominal value of λ depends on the radius in the disc, and on density profile as well, which is not acceptable.

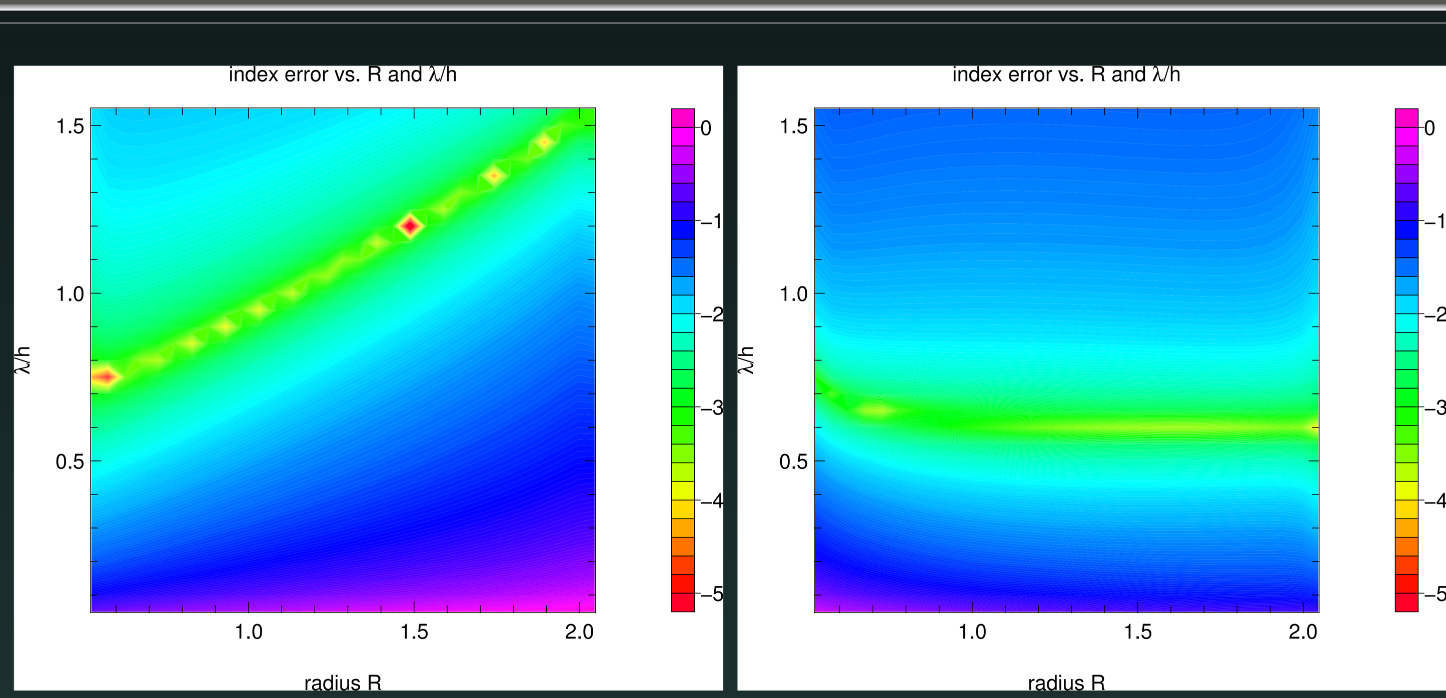


Figure 1: Relative error (log. scale) on potential values when computing the potential in a flat homogenous disc (*left*) and in a flared power-law disc (*right*) when using the softened potential with $\lambda/h = cst$. Parameters and setup: disc inner edge at 0.5; discretization into 32×64 cells in the (R, θ)-plane; regular spacing; $h \propto a$ and mass density $\rho \propto a^{-2.5}$ for the flared power-law disc.

2. λ can be an imaginary number !

According to the Newtonian theory, the potential of a cylindrical sector, as depicted in figure 2, is given by the triple integral

$$\psi_{\rm cell}(\vec{r}) = -\frac{Gm}{V}\int_{z_0-h}^{z_0+h} dz \int_{a_0-\frac{1}{2}\Delta a}^{a_0+\frac{1}{2}\Delta a} da \int_{\theta_0'-\frac{1}{2}\Delta\theta'}^{\theta_0'+\frac{1}{2}\Delta\theta'} \frac{a d\theta'}{|\vec{r} - \vec{r}'|}. \quad (3)$$

So, λ reproduces exactly the Newtonian potential of the cell if

$$\psi_{\rm Plum.}(\vec{r}; \lambda) - \psi_{\rm cell}(\vec{r}) = 0, \qquad \text{i.e.} \qquad \lambda^2 = \left(\frac{Gm}{\psi_{\rm cell}}\right)^2 - |\vec{r} - \vec{r}_0'|^2, \quad (4)$$

everywhere in space. We notice that $\lambda^2 < 0$ is possible.

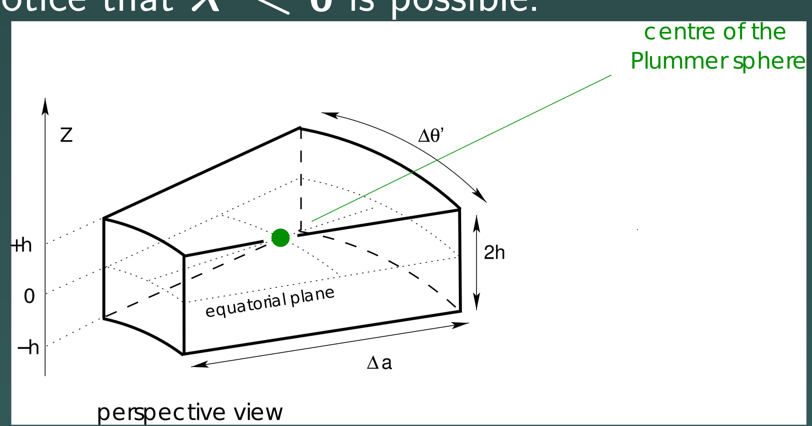


Figure 2: Cylindrical cell (3D-view and projection in the midplane), centre of the Plummer sphere and associated notations. Points A to E mark the centre of neighbouring cells.

We have computed λ from Eq.(4) using the contour integral given by [Huré et al., 2014], see figure on the right.
It turns out that λ can be an **imaginary number** !

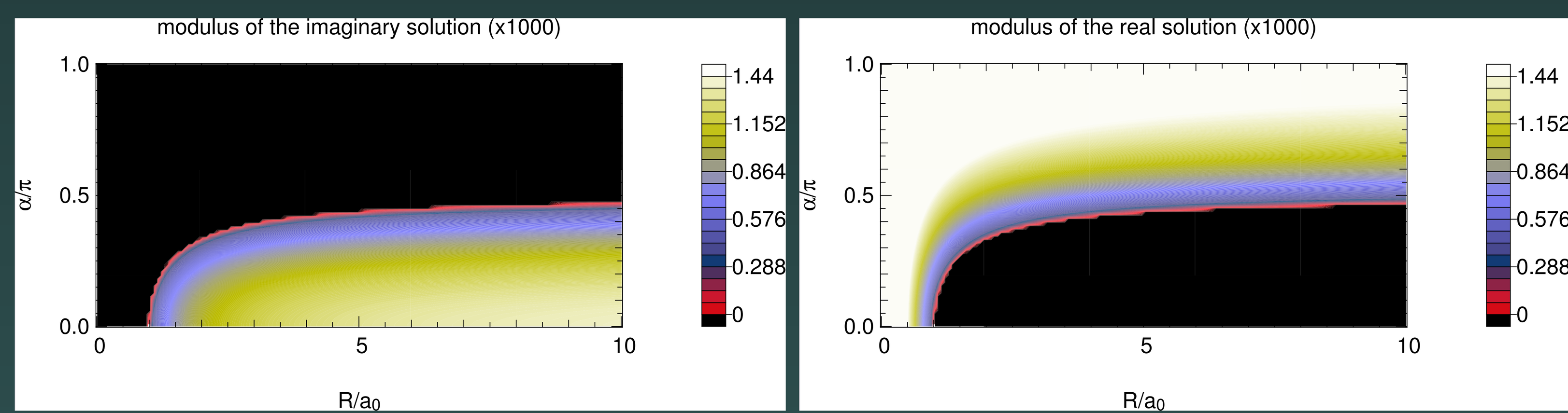


Figure 3: Value of λ computed in the equatorial plane from Eq.(4): imaginary solutions (*left*) and real solutions (*right*). The coordinates of the cell's centre are $R/a_0 = 1$, $\alpha = 0$ and z_0.

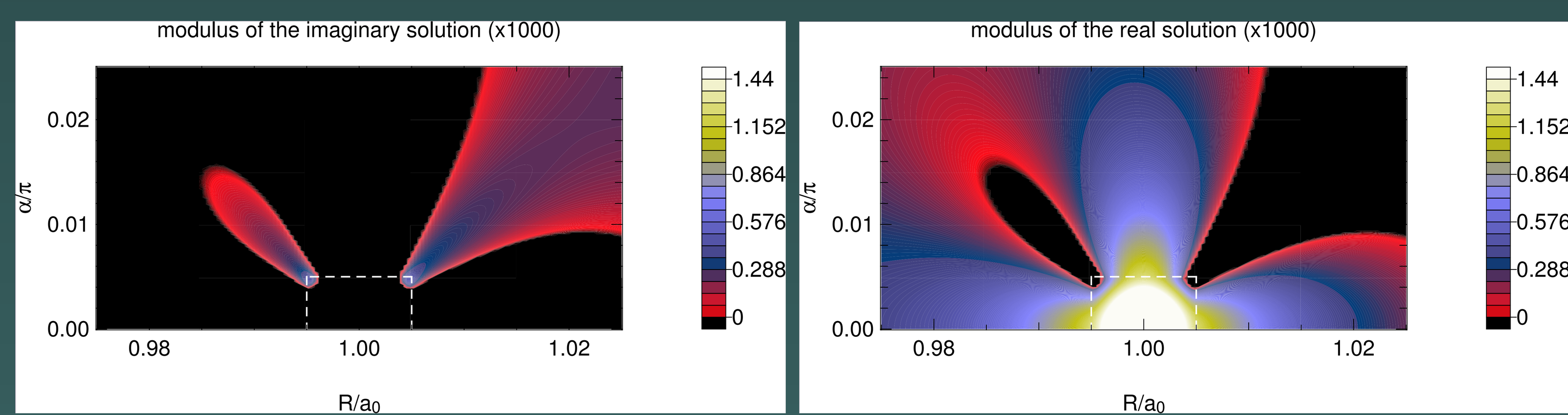


Figure 4: Same legend **and same color code** as for Fig. 3, but zoomed around the **numerical cell** (boundary in dashed white line).

3. λ at long-range from a dipolar expansion

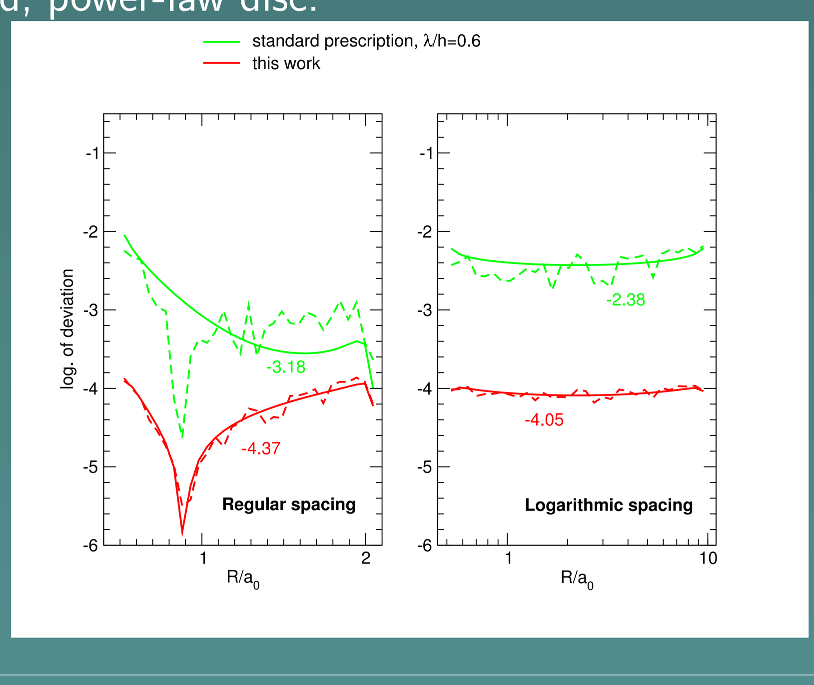
We can expand the kernel $1/|\vec{r} - \vec{r}'|$ over $\alpha = \theta - \theta_0'$ before performing the integrations in Eq.(3). We then focuse on the long-range behavior of $\psi_{\rm cell}$ in Eq.(4). After some tedious calculus, we obtain the following approximation for λ:

$$\frac{\lambda^2}{4a_0 R} \approx \frac{\Delta\theta'^2}{48}\cos\alpha - \frac{1}{24}\left(\frac{\Delta a}{a_0}\right)^2 \left|\cos\alpha - \frac{3a_0\left(\zeta_0^2 + R^2\sin^2\alpha\right)}{2|\vec{r} - \vec{r}_0'|^2}\right| + \frac{h^2}{12a_0 R} \equiv \bar{\lambda}^2 \quad (5)$$

We conclude that λ is not proportional to h, as often considered.

Conclusion & perspectives

With such an improved prescription, the error on potential values is reduced by $2 - 3$ orders of magnitude typically with respect to the standard prescription where $\lambda \propto h$. This is illustrated in the figure below for a flared, power-law disc.



To conclude, we show that

- the standard prescription $\lambda \propto h$ is not the nominal choice.
- the appropriate softening length can be an imaginary number; this corresponds to a point mass potential weaker than that of the cylindrical cell.
- λ is a complicated function of space and cell's geometry. Collective effects show λ depends on how the disc is discretized (radial, azimuthal and vertical sampling).
- it is possible to determine an approximation for the softening length valid at long-range. Thus, this formula enables to mimic the Newtonian potential of a cylindrical from the Plummer potential, at long-range.

More details are given in Huré and Trova [2014].
This work can be continued in several ways. It could be interesting to produce a complete analytical value of λ (valid at short-range too), which is tricky, and to see how this new prescription impacts on hydrodynamical simulations.

References

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