Galactic Rotation Curve with and without Dark Matter, but with Excel

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What we want to simulate

We would like to interpret the observational data on the rotation curve of a galaxy, such as our Milky Way, by computing the speed for circular orbits as a function of the distance from the galactic centre. This speed depends on all the mass within a sphere of that radius. Since a spiral galaxy is composed of a central spherical bulge, a disk of gas and stars, a spherical halo of stars, and perhaps a spherical halo of dark matter, we have to sum up the contributions from each component.

This calculation is quite similar to that of a system evolving in time: we compute the mass inside a certain radius by starting at the centre and then summing up all the contributions to the mass in the subsequent spherical shell. Instead of a evolution in time, we progress from the centre outward!

Having done the computation, we compare with observed data, and play with the parameters of the galaxy until we can best match the rotation curve. However, there is a general problem: the rotational speed is rather constant with radius … Our standard explanation is to postulate the existence of a spherical halo of dark matter, in which the visible galaxy sits, and you can determine the mass of that dark halo.

However, so far we have no independent evidence for the existence of that unseen dark matter, and therefore people have considered alternative explanations. One is MOND for MOdified Newtonian Dynamics, which assumes that Newton's laws do not apply for the large scales, long times, and small accelerations that one finds in the Universe as they do in the minute context of our labs and the Solar System. You can explore how well this matches the galactic rotation curve.

The equations

The basic equation is quite trivial: it tells us how the mass m(r) of a sphere of radius r increases with radius, given the local mass density ρ(r):

$$dm/dr = 4 \pi r^2 \rho \qquad (1)$$

How does the density r depend on radius in a galaxy? The matter is not distributed like a uniformly filled sphere, because the shape into which stars assemble is determined by the gravity of the stellar population itself, and by its geometrical shape … We shall use some reasonable recipes:

- For the bulge and the halo, we can use a "Plummer sphere":

  $$\rho(r) = M / (4\pi a^3) * (1 + (r/a)^2)^{-5/2} \qquad (2)$$

  here M is the total mass in this component, and a is the scale radius, which determines the size into which the matter is concentrated. The density is largest in the centre, it falls off towards the exterior, and a can be regarded as the radius within which most of the mass is concentrated
- There is also a sphere-shaped halo of old stars and globular clusters, which we can approximate by:

  $$\rho(r) = M / 109 * (a + r)^{-3.1} \qquad (3)$$

  with the total mass M, and a scale radius a=0.5 kpc; the density of the stellar halo falls off with radius more rapidly than in a Plummer sphere! This formula is a simple fit to the star count data from our Milky Way.
- For the disk, we shall use the approximation of a thin disk, whose projected density Σ(r) drops of exponentially with radius

  $$\Sigma(r) = M / (2\pi a^2) * \exp(-r/a) \qquad (4)$$

  again, M is the total mass of this component and a is the radial scale length, which determines how fast the density drops off. Since the disk has a non-spherical geometry, its contribution to the mass m(r) inside radius r is computed somewhat differently:

  $$dm/dr = 2 \pi r \Sigma \qquad (5)$$

How to solve the equations

We start at the centre r=0, where we know the mass m(0) = 0. We compute for that radius from (2), (3), and (4) the densities of all components. Next we consider the radius being increased by a small step Δr and we compute from the densities using (1) and (5) for each component the increment of the masses inside radius r. Then we compute the mass inside that larger radius by:

$$m(r+\Delta r) = m(r) + dm/dr * \Delta r$$

At the new radius we repeat the procedure …
Since the rate dm/dr has a different formula for spherical and disk-like components, you can either compute the masses m(r) for each component separately and then add them up to the total mass m(r) inside radius r, or combine the equations (1) and (5) …
A galaxy has a diameter of about 30 kpc. Choose the radial step Δr suitably small, say 0.1 kpc, so that you get nice, smooth curves in the plots.

The circular speed

At every radius, we then have the total mass m(r) inside radius r. Next, we compute the gravitational acceleration due to this mass

$$g(r) = G\, m(r)/r^2$$

and from this we compute the orbital speed by noting that the gravitational acceleration is balanced by the centrifugal acceleration v^2/r for a circular orbit, hence

$$v_{circ}(r) = \sqrt{g(r) * r}$$

Units and constants

It's best to work in SI units: m, kg, s … so we have to convert the convenient numbers based on the units used in astronomy. Here are the respective constants in SI units:

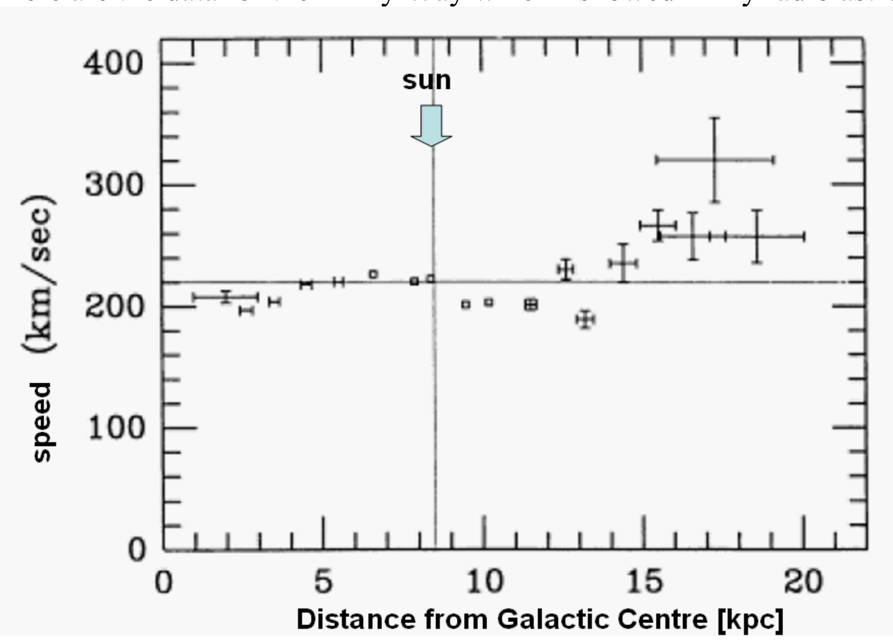
Gravitational constant $G = 6.67259\ 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$

1 solar mass $= 1.989\ 10^{30}$ kg

kilo parsec (about 3000 light years) 1 kpc $= 3.086\ 10^{19}$ m

The observed rotation curve

Here are the data for the Milky Way which I showed in my radio astronomy lecture:



r [kpc]	v [km/s]
2	210
2.5	198
3.5	210
4.5	218
5.5	220
6.7	213
7	220
7.5	222
9.5	200
10.3	203
11.5	200
12.5	230
13.3	190
14.5	240
15.5	270
16.5	260
17.5	320
18.5	260

Galactic parameters

For the Milky Way, we know the mass and the scale lengths of the visible components:

- The bulge: $M_B = 1.3\ 10^{10}$ solar masses, a = 0.4 kpc
- The disk : $M_D = 6.5\ 10^{10}$ solar masses, a = 4 kpc
- The (visible) halo: $M_H = 10^9$ solar masses, a = 0.5 kpc

With dark matter…

- Which values for the total mass and the scale parameter $a_{Dark\ Halo}$ for a spherical halo would be required to match the observed data?

… and without: The MOND alternative

One common formulation of MOdified Newtonian Dynamics assumes the standard (Newtonian) gravitational acceleration g_N (which we had computed as above) is modified in this way

$$g * \mu(g/a_0) = g_N \qquad (6)$$

where μ(x) is a function which is equal to 1 for large values of the argument, but equal to x for small values. This function is not yet precisely known, it merely represents our first guess of what a modified gravity formula could be! There is no physical theory for that …If we take a simple form, such as μ(x) = x/(1+x), we can resolve Equation (6) analytically and get for the modified gravitational acceleration:

$$g = [g_N + \sqrt{g_N * (g_N + a_0)}\ ] / 2 \qquad (7)$$

- Use this modified version, and see what value of the constant a_0 you need to explain the Milky Way's rotation curve, using just the bulge and the disk, whose matter is visible …

The constant a_0 is about 10^{-10} m/s² and represents the acceleration above which our normal Newtonian laws apply, but below which the gravitational law is modified …

Caveat:

However, my description here is nothing but a **brutal simplification**! True calculations with MOND are much more complicated … but this gives you a bit of taste of what people are exploring as alternative explanations to get rid of that nasty 'dark' matter'!