Astrophysics with the Computer: Circular and escape velocities from a galactic disk

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1 Astrophysics

The movement of stars within a galaxy is governed by the gravitational potential in the galaxy which is produced by all the visible and nonvisible matter in the galaxy. So the stars of the disk, of the bulge, and of the halo as well as the gas in the disk influence the kinematics of any star, say in the disk.
The gravitational field produced by the disk stars is symmetric with respect to the plane, and also axisymmetric with respect to the rotational axis. Here we shall consider the gravitational field in the galactic plane only, which will thus depend only on the distance from the galactic centre. How will be this radial dependence and how will this affect the movements of stars?
If we watch a star far away from the galactic centre, the gravitational attraction can well be approximated by assuming that all the mass of the galaxy is concentrated in the galactic centre. Hence the potential will be that of a point mass, like that of the sun; the star will move like a planet around the sun, in an orbit that has the shape of a conic section. The escape velocity from a given radius will be a certain multiple of the circular velocity.
But what happens, if we are nearer to the galactic disk or even within it? What are the circular and escape velocities? Is their ratio still the same?
From observations it is known that the circular velocity in spiral galaxies is pretty independent of the distance from the galactic centre. A single point mass in the centre would not be able to account for that – and neither is the visible matter in the disk and the bulge. How do the rotation curves look like which are due to example the disk only? How must the dark matter be distributed so that the overall rotation curve is as flat as observed?

2 The Equations

We shall treat the disk as an infinitely thin disk, thus it shall be described by the radial dependence of the surface density Σ which for a simple exponential disk model is like:

$$\Sigma(r) = \Sigma_0 \exp(-r/R) \qquad (1)$$

with a radial scale length R of about 4 kpc for the Milky Way. The central surface density Σ_0 is obtained from the total mass

$$M_{\rm disk} = 2\pi \int_0^\infty \Sigma(r) r dr \qquad (2)$$

The circular velocity at some radius r is determined from the condition that there the centrifugal acceleration balances the gravitational attraction at radius r

$$\frac{v_{\rm circ}^2(r)}{r} = a_{\rm grav}(r) \tag{3}$$

You may say that this atrraction is produced by all the mass $M_{\rm r}(r)$ inside radius r

$$a_{\rm grav}(r) = -\frac{GM_{\rm r}(r)}{r^2} \tag{4}$$

and since the disk is axisymmetric, we can easily compute this mass as

$$M_{\rm r}(r) = 2\pi \int_0^r \Sigma(r) r dr \tag{5}$$

This would be nice, but: ... but is that really true?

This argument holds only for **spherically symmetric** mass distributions! In the general case, we must compute the gravitational acceleration from the potential Φ

$$a_{\rm grav}(r) = -\frac{\partial \Phi(r,z)}{\partial r} \tag{6}$$

(Perhaps you can show analytically whether the argument might also be true in our axisymmetric case?). Likewise, we need the potential to get the escape velocity which is obtained from the condition that the kinetic energy equals the local potential:

$$\frac{v_{\rm esc}^2(r)}{2} = -\Phi(r) \tag{7}$$

The gravitational potential is the work per unit mass necessary to move a star from its position to infinity, which must be done against the gravitational attraction. Because the gravitational field can be described by a potential, we may choose any path from the initial position to infinity! For convenience, we shall take a path in the galactic plane, going outwards in radial direction:

$$\Phi(r,0) = -\int_r^\infty a_{\rm grav,r}(r',0) dr' \tag{8}$$

where we have to compute the acceleration $a_{\rm grav,r}(r,z)$ in radial direction at radius r and height z above the plane. This acceleration is the sum of all contributions by all the masses in the disk. Note that the acceleration is the vector sum of these contributions, but we need only the x-component, if we chose the integration path to follow along the x-axis. If we chose a cartesian coordinate system centered in the galactic centre, we need to compute

$$a_x(r,0) = G \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\Sigma(x',y')}{d^2} \frac{x-x'}{d} dx' dy' \tag{9}$$

where d is the distance between point $(x',y',0)$ and the position $(x=r, y=0, z=0)$.

3 Numerical Methods

For the integrations, you can use the trapezoidal rule, or Simpson's rule or any other method you favour. Of course, we must make sure that the method as well as the integration steps are chosen in such a way, that the errors on the results are *sufficiently* small. For our comparison of circular and escape velocities, an overall accuracy of about 1 percent should be achieved.

One has to choose finite values for the integration limits, because neither the number nor the limit ∞ is represented on the computer. Since the density of the disk decreases outward, we can choose a radial distance far enough so that the contributions to the integral are less than the error requested. This may be done by trial and error, but also by some estimate for the integration error. Note that if you are far enough from the centre, you can estimate the potential by supposing that all the mass is concentrated as a point in the centre.

Also, we have to choose a coordinate system, say a cartesian system centered in the galactic centre, with the $x - -y$ plane in the galactic plane. Then we only need to chose integration steps Δx, Δy, and Δr and the integration intervals, and then we compute Eqn. 9. You may well encounter serious problems in obtaining good results, or it may take long computation times to have stable results ... the only thing to avoid is that the distance d should never become zero!

The use of a cartesian coordinate system in an axisymmetric problem is not overly clever and efficient. No serious problems will occur as long as the test point r is outside the disk ... but when we want to compute the acceleration on a position within the disk, we may run into serious trouble: the accelerations from points in the close neighbourhood will be large and they might dominate the results! The acceleration diverges for zero distances: In Fig. 1 we plot the contributions to the acceleration in x-direction for a test star at $x = 3$ as a function of x-position in the galaxy. One sees that the maximum contribution from the galactic centre is dwarfed by the large accelerations close to the test star, but also that this nearby contribution is both positive and negative, thus it will cancel out ... Let us look more closely at the situation: in a very very small neighbourhood around the test star the accelerations will be very large, but since the surface density is practically constant, the accelerations will have the same value but come from all azimutal directions. Thus, the integral over a ring around the test star will give a zero net acceleration – this is why a polar coordinate system will be much better; check that the numerical evaluation of this integral gives 'zero' with acceptable accuracy.

One good way to compute the total acceleration is to use a polar (in 3D: spherical) coordinate system **centered on the position of the test star**, thus not on the galactic centre. The azimutal integration in the close neighbourhood gives very small contributions – despite the large individual accelerations – and the radial integration can start with some suitably chosen small distance ...

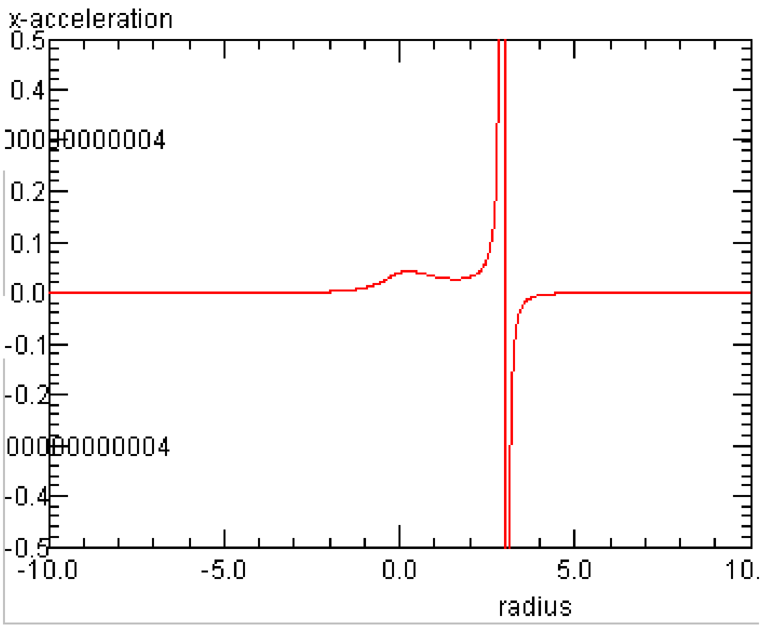

Figure 1: Contributions of the various parts of the galaxy to the acceleration in x-direction at the position $x = 3$ of a test star.

4 Test cases

The simplest system is a point mass M at the centre. The potential is simply

$$\Phi(r) = -\frac{GM}{r} \tag{10}$$

The escape velocity follows directly. From this one obtains the centripetal acceleration by taking the radial derivative. The circular velocity follows from the equality of centripetal and centrifugal forces.

A description often used for spherical self-gravitating systems of stars is the Plummer sphere. It has the potential

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + a^2}} \tag{11}$$

and it is produced by the gravitation of stars distributed in space with the density

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1 + (\frac{r}{a})^2\right)^{-5/2} \tag{12}$$

where r is the distance to the centre. The constant a is the characteristic radius of the density distribution. This case is a good test case for a 1-D integration over the radial coordinate r.

For axisymmetric systems (elliptical galaxy, disk of a spiral galaxy) one can generalize this model and obtain the Toomre-Kuzmin disk model:

$$\Phi(\omega, z) = -\frac{GM}{\sqrt{\omega^2 + (a + |z|)^2}} \tag{13}$$

where ω shall denote the radial distance to the axis of symmetry. This potential is produced by stars which are distributed in the galactic plane, having a *surface density*

$$\Sigma(\omega) = \int \rho(\omega, z) dz = \frac{M}{2\pi a^2} \left(1 + (\frac{r}{a})^2\right)^{-3/2} \tag{14}$$

Your program should be able to compute the potential, if you give the surface density profile!

A further generalization gives the Miyamoto-Nagai potential

$$\Phi(\omega, z) = -\frac{GM}{\sqrt{\omega^2 + (a + \sqrt{b^2 + z^2})^2}} \tag{15}$$

which is the result of the mass distribution

$$\rho(\omega, z) = \frac{Mb^2}{4\pi a^2} \frac{a\omega^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})}{[\omega^2 + (a + 3\sqrt{z^2 + b^2})^2]^{5/2}(z^2 + b^2)^{3/2}} \tag{16}$$

You can use this to test your 3-D version of integration over the given mass distribution.

5 How to do it

You should start with testing a method of numerical integration (of your choice) by computing a one-dimensional integral over a number of suitable test functions. In this way, you obtain experience of how the integration steps and the integration intervals should be chosen, in order to get the desired accuracy.

Then you might try the Plummer sphere, since it requires an integration in one dimension only.

For the 2-D case, it is worth trying out the simple stupid integration with the cartesian coordinates ...

Formulate the integration method for the polar coordinate system centered on the test star position, and make it work. Exploit the symmetries of the stellar disk to avoid unnecessary calculations and to speed up the computations.

To test rigorously your program, you derive the analytical solutions from the potentials of the test cases, and compare with your numerical results. It is also a good idea to test with a spherically symmetric case.

The eventual task is to compute the circular and escape velocities as functions of radial distance from the galactic centre for an exponential disk, say of a total mass of 10^{11} $M_{\odot}$and with radial scale length of 4 kpc. There may be in the literature the analytical solution for the exponential disk... play with the parameters ...

Further possibilities are:

- how large are the errors in the rotation curve, if one makes our initial mistake to compute the circular velocity at radius r by considering simply the mass inside the circle (or sphere) of this r?

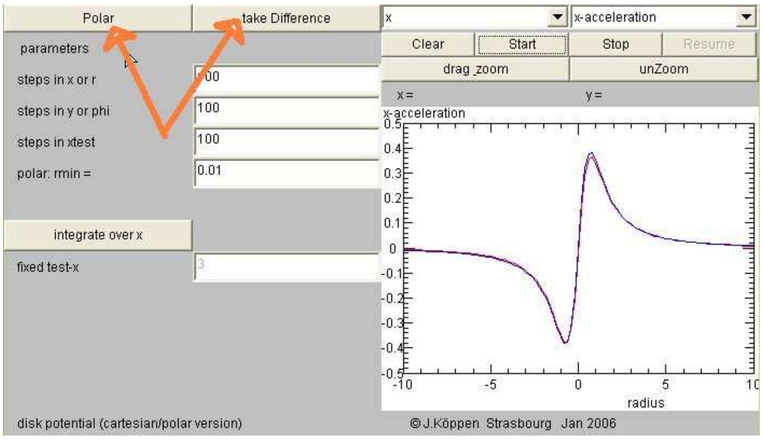

Figure 2: The potential of a Kuzmin disk, computed with polar coordinates and integrating over surface densities reduced by the surface density at the test star.

- what happens, if one adds a point mass or a Plummer sphere – representing the bulge – in the centre?
- how does the rotation curve for this galaxy look like? Is is "flat" at a constant velocity of 200 km/s, as one observes in normal spiral galaxies?
- add a Plummer sphere of dark matter to make the rotation curve flat...
- ... or add a spherical dark matter halo with $\rho(r) \propto r^{-3}$, for which one can also compute analytically the potential
- try to fit the rotation curve of our Milky Way or another spiral galaxy. You'll find the data in the literature.
- extend the numerical calculations into three dimensions to compute thick disks $\rho(r,z) \propto \exp(-r/R)\exp(-|z|/H)$ in cylindrical coordinates. H is the vertical scale height of the disk, about 250 pc for the stars. How large are the differences in the rotation curve and escape velocities between a full 3-D calculation with a disk of finite thickness and a 2-D calculation assuming an infinitely thin disk?

6 Improvements

The accuracy of the results can further be improved, if one integrates not over the density $\Sigma(x,y)$ but over the density reduced by the density at the position of the test star: $\Sigma(x,y) - \Sigma(x_t, y_t)$. This gives smaller numbers for the functions over which we integrate, and therefore the cancellation of positive and negative terms is better.

7 Literature

The books "Galactic Dynamics" by Binney and Tremaine, "Galactic Astronomy" by Binney and Mihalas are available in the MS2 library.

A test version of a Java applet for the Kuzmin disk - but still without any explanations – is available at
`http://astro.u-strasbg.fr/~koppen/galpot/test.html`